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INTRODUCTORY MATHEMATICAL METHODS FOR ECONOMICS

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Introductory Mathematical Methods for Economics

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Discipline Specific Course – 2
INTRODUCTORY MATHEMATICAL METHODS FOR ECONOMICS
Study Material : Lesson 1-13

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LESSON 1

LOGIC AND PROOF

STRUCTURE

1.1 LEARNING OBJECTIVES

After reading this lesson, students will be familiar with the language of mathematical Economics. Students will learn:

1. Statement and its Negation
2. Truth Tables with conjunction (and) ; disjunction (or); and implication
3. Use of conditions if, only if, if and only if.
4. Use of necessary and sufficient condition
5. Quantifiers
6. Concept of Proof which includes direct proof, proofs by contradiction, induction and proof by using Contrapositive.

1.2 INTRODUCTION

This chapter will familiarize you with the language of Mathematics and this is crucial to develop your understanding of the Microeconomics, Macroeconomics and other important Core subjects further to be learnt in the Course.

This Lesson includes various essential concepts that will help in building the conceptual understanding of the subject and will also help in better understanding of economic theory and concepts.

This chapter primarily includes Truth Tables for negation, Conjunction i.e. 'and', disjunction 'or', and Implication. Necessary and sufficient conditions and Quantifiers are explained in detail. This chapter includes various methods of proofs like proof by contradiction, contrapositive, direct-and indirect method. After reading this chapter you will be familiar with the concept of logic and proof.

1.3 STATEMENT AND TRUTH TABLES

In English language course, you must have studied different kinds of sentences such as assertive, imperative, exclamatory and interrogative. In Mathematics, only assertive sentences (i.e., sentences stating any fact) are regarded as Statement. For example, four is an even number. This is an assertive sentence and therefore can be regarded as a statement in Mathematics.

1.3.1 Proposition

A proposition is a statement that can be either true or false; it must be one or the other, and it cannot be both.

For example: Radha is a doctor is a proposition as this statement can either be true or false.

"Shut up!" is not a proposition since it cannot be regarded as true or false. Similarly, how far is school?" is not a proposition since it will have subjective answers.

$x + 3 = 3x$ is not a proposition. If $x + 3 = 3x$, where $x = 3/2$ is a proposition.

Some other important terms that must be learnt are as follows:

Axioms: An axiom is a basic assumption about a mathematical situation. Axioms can be considered facts that do not need to be proved or they can be used in definitions.

Hypothesis: A hypothesis is a proposition that is consistent with known data but has been neither verified nor shown to be false.

Conclusion: Conclusion is final result of reasoning, research or hypothesis.

1.3.2 Negation

A Mathematical Statement can be true or false Negation of a Statement A is not A, Negation of a statement B is not B. Not is represented by '¬' symbol.

Now let us learn the truth Tables. Truth Tables are the important Part for learning logic. Truth Table summarizes all the possibilities of truth and false in a table.

The Truth Table for Negation is represented as follows.

A	¬A
T	F
F	T

Table 1: Truth Table for Negation

In the above-mentioned table T means true and F means false. This table represents that if A is true then ¬A (not A) is False and if A is false then ¬A (not A) is true. We consider the first column as Input and second column as output. So, if we input True. We get Output as false for not A.

1.3.3 Conjunction

In English language connectors connect two statements. Similarly in Mathematics connectors like 'or' and 'and' connect two statements together. Let us study 'and' connector. It is represented with '^' symbol. Let us assume two statements A and B. We will have four possibilities i.e. A is True, B is True; A is False, B is True; A is False B is False; A is true, B is False.

A	B	A^B

T	T	T
F	T	F
T	F	F
F	F	F

Table 2: Truth Table for Conjunction

In Table 2 the first two columns represent Input and A^B represent output. According to this Truth Table if A and B both are true only then A^B will be true. If either of them is false, then A^ B will be False. Note that 'and' is also known as conjunction

1.3.4 Disjunction

Let us study disjunction or in other Words 'or". Disjunction is represented by 'v' symbol and truth Tables for Disjunction is as follows.

A	B	A v B
T	F	T
F	T	T
T	T	T
F	F	F

Table 3: Truth Table for Disjunction

As opposed to conjunction in disjunction if either of the Statement True then A v B is True. Now we can use truth Table to solve complicated expressions

A	B	¬B	Av¬B
F	F	T	T
T	T	F	T
T	F	T	T
F	T	F	F

Table 4: Truth Table for Av¬B

1.3.5 Implication

We often come across some statement stating that if A is true then, Statement B is also True. To understand these Statement, we have to study the truth table for Implication.

A	B	A→B
T	T	T
T	F	F
F	T	T
F	F	T

Table 5: Truth Table for Implication

In Table 5 statement A is Hypothesis and Statement B is conclusion. Only the second row goes against the Basic definition of Implication because if A is true then B should be true to get true. If A is True and B is false, then statement cannot be true as with true Hypotheses we cannot get false Conclusion. If A is false & B is true, then A→B is true. If A is false and B is false, then A→B is true. Since the hypothesis is false then conclusion is also false so, this makes a true implication.
Suppose X and Y are two propositions such that whenever X is true then Y is necessarily true.

IF and IFF condition

It can be written as X→Y i.e., "X implies Y" or "If X then Y" or "Y is a consequence of X". "→" is an implication arrow that shows the direction of the logical implication. So, if condition or "only if" represents implication. Let there be two propositions, X and Y. X states that you are born in a country; Y states that you are a citizen of that country. If X then Y is true. If you are born in a country, then you are a citizen of that country. So, X→Y is true. Here Y→X is not true, i.e., to become a citizen of the country it is not necessary to be born in that country. So, "only if" condition states represent X→Y to be true. X→Y can also be expressed that "X only if Y", "Y if X" & "Y is an implication of X"

If Y→X, X→Y then we can write both implication as X↔Y, it is known as "if and only if" or also known as "iff" condition. Let there be two proposition X and Y, where proposition X states that two lines are parallel and proposition Y states that two lines are not intersecting. So, Y→X, and X→Y are true. Therefore "iff" or "if and only if" condition is satisfied. It means that "X is equivalent to Y", because we have X if Y and Y if X "↔" this is an equivalent arrow. It is also represented by always equal to symbol i.e., "≡". For example, ab = 0 ↔ (a = 0 or b = 0) here we have used equivalent sign because a = 0 or b = 0 implies ab = 0 and ab = 0 implies a = 0 or b = 0.

IN-TEXT QUESTION

1. Construct a truth table for ¬P V ¬Q
2. Construct a truth table for ¬P^P

1.4 NECESSARY AND SUFFICIENT CONDITION

Sufficient condition

Consider the following statement.

Rain occurs than rivers flow

The occurrence of rain is sufficient condition for the river to flow.

So, if Rain Occurs is considered to be statement A and River flows is considered to be Statement B.

So, if generalize the sufficient condition then A→B which means A is the sufficient condition for B.

Necessary Condition

Please consider the following example.

Glowing Bulb requires electricity.

So, if we generalize this example so, consider glowing bulb as A'. Consider electricity as Statement B'.

So, B' is a necessary condition for A'

This means, A'→B'

Note that B' is Necessary Condition for A if, A'→B' and A is sufficient condition for B if A→B

1.5 QUANTIFIER

Quantifiers define quantity as all, some. There are two kinds of quantifiers universal and existential.

1.5.1 Universal Quantifier: "for all" is the phase for this quantifier. It can be denoted by "∀"

For Example, any $X \in Z$ This $x^2 \geq 0$. This statement says that square of an integer number is non-negative number. Here we can use "for all"

1.5.2 Existential Quantifiers. "There exist" is the phase for this quantifier. It can be denoted by quartet. the phrase for be donated by '∃'. Belongs to is represented through the symbol "∈". If we say that p belongs to natural number, then we can write $p \in N$. here symbol "∈" represents belong to and natural numbers are represented through N.

For example

$\exists\ p \in N$ such that $p^2 = 9$. This Statement is true as p=3 satisfy that $p^2 = 9$.

We can combine the two quantifier that we discussed above

"For all $p \in N$ there exists $q \in N$ such that $q \geq p$

We can write this as $\forall\ p \in N\ \exists\ q \in N\ | q \geq p$"

1.6 PROOFS

Proof provides a guarantee for the correction of solution. They provide justification and help to convince the truth of the statement. We have used proofs extensively in Mathematics. It makes the statement easy to understand and convinces the reliability of the statement. It is important to know the different ways to prove a statement. There are basically four different techniques that can be used to prove a statement i.e., as follows:

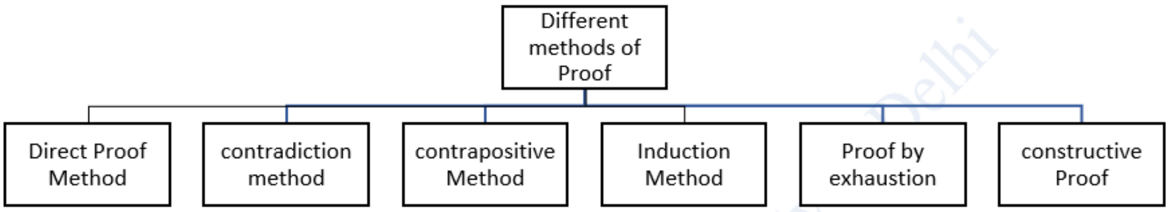

Figure1: Different methods of Proof

1.6.1 Direct Proof Method

Direct Proof method is simply used to show that P imply Q is true or in other words it states that If P then Q statement is true.

For example: for all Y belongs to Integer, if Y is odd then 3Y+5 is even. So, here hypothesis is Y is odd and Implication is 3Y +5 is even. So, by direct proof method

If Y is an odd Integer

Then $Y = 2m+1$ for some integer m and $2m + 1$ will be odd. If $m = 1$ then $2x1 + 1 = 3$ which is odd. If $m = 2$ then $2x2 + 1 = 5$ which is odd. So, a number Y could be represented with the help of a function.

$Y = 2m+1$ where m can take any positive Integer value.

So $3Y + 5 = 3 * (2m + 1) + 5$ (by substituting the value of Y in 3Y+5)

$= 6m + 3 + 5$

$= 6m + 8$

$= 2(3m + 4)$

= 2l

We have taken l = 3m + 4

As we know any positive integer multiplied by 2 gives an even number therefore it is proved that if Y is odd then 3Y+5 is even number.

1.6.2 Proof by Contradiction Method

In this method we assume the statement to be false. So, by using P implication Q we will assume Q to be false and will try to prove that ¬Q is true and if the Q is true then the contradiction will arrive while solving for ¬Q to be true. Thereby we will prove that ¬Q is false, and Q is true by following the method of contradiction.

Let us prove by contradiction that if 5n + 4 is odd then n is odd. So, here hypothesis i.e. P is 5n + 4 and conclusion is n is odd.

The negation of n is odd is n is even. If we assume n to be even, then we can write

$n = 2m$ (where m is any Integer)

So, by substituting $n = 2m$ in $5n + 4$ we get

$5n + 4 = 5(2m) + 4$

$5n + 4 = 10m + 20$

$5n + 4 = 2(5m + 10)$

Let = $5m + 10 = l$

So, $5n + 4 = 2l$

2l is even, and we are getting contradictory result or in other words if n is even true statement, then $5n + 4$ must be odd, but by solving we found $5n + 4$ to be even which means that n is even in itself is a wrong assumption and n is odd is true.

1.6.3 Proof by Contrapositive

Proof by contrapositive is also known as indirect proof Method, before that it is necessary to show that P→Q is equivalent to ¬Q→¬P

P	Q	P→Q	¬P	¬Q	¬Q→¬P
T	F	F	T	F	F
T	T	T	F	F	T

F	F	T	T	T	T
F	T	T	F	T	T

Table 6: Truth Table to show equivalence of ¬Q→¬P and P→Q

As you can clearly see from (Table 6) that P→Q is equivalent to ¬Q→¬P. So, by adopting the same technique we will prove a statement by contrapositive method by taking the same example. If $5n + 4$ is odd, then n is odd.

Where hypothesis is P is $5n + 4$ is odd and conclusion Q is n is odd. But, according to the method we have to follow ¬Q→¬P

So, ¬Q is n is even then $n = 2m$ for m belongs to Integer

$5n + 4 = 5\ (2m) + 4$ (by substituting $n = 2m$ in $5n + 4$)

$5n + 4 = 10m + 20$

$5n + 4 = 2(5m + 10)$

$5n + 4 = 2l$ (where $l = 5m + 10$)

So, 2l is even

So, n is even then $5n + 4$ is even, so by adopting the method of indirect proof we have hence proved the statement

IN-TEXT QUESTION

3. If n is an Integer and $9n + 8$ is odd, then n is odd proved by contrapositive Method.
4. What are the different methods of proof?
5. Is contrapositive being a method of proof?

1.6.4 Proof by Inductions

As the world 'induction' suggest that it gives rise to something or by simple example if many people are standing in a queue and if we push one person, it knocks second and that in turn knocks the third person and so on, so this is the process of Induction.

So, we Proof kth statement to be true then (k+1) statement will also be true.

So, If kth → $(k + 1)$th is true.

This method follows some steps that is $f\,(1)$ is true which is regarded as basic or initial step then we check for $f(k)$ which is called Inductive step. We assume $f(k)$ to be true which is regarded as inductive hypothesis to prove $(k + 1)$ to be true.

1.6.5 Proof by Exhaustion Method

In this method a proposition is divided into many cases, and we have to check each of the cases and respective domains. If there are two proposition X and Y and X implies Y, then we have to divide X in n finite cases i.e., $X_1, X_2, X_3 \ldots X_n$. By proving $X_1 \rightarrow Y$, $X_2 \rightarrow Y$, $\ldots X_n \rightarrow Y$, we prove $X \rightarrow Y$.

For example: Proof that if n is an integer, then $n \leq n^2$.
Solution: So, we have to check all the possible cases.
Case 1: $n \leq -1$ so $n^2 > o$, it follows $n \leq n^2$
Case 2: $n = 0$, it follows $n \leq n^2$
Case 3: $n \geq 1$, it follows $n \leq n^2$

1.6.6 Constructive Proof

Constructive proof helps to prove logic by following a specific mathematical algorithm to arrive at a logical conclusion.

For example: show there exist a solution to $x^2 + y^2 = 50$ using natural numbers.
Proof: let $x = 7$ and $y = 1$, $x^2 + y^2 = 7^2 + 1^2 = 50$ is a constructive proof.

1.7 SUMMARY

This chapter has familiarized you with basic concepts related to logic and proof. This unit has familiarized you with different kinds of Truth Tables like truth table for conjunction, disjunction, and implication. This chapter has explained to you the concept of necessary and sufficient Condition. Two kinds of Quantifiers which is universal and existential are discussed in the chapter. We have finally discussed four important methods of proof i.e., direct proof method; Proof by induction; proof by contrapositive; proof by contradiction with examples to create better understanding of the concept. This will help you to think about economic theories in a much better way.

1.8 ANSWERS TO IN-TEXT QUESTIONS

Solution 1:

P	Q	¬P	¬Q	¬P V ¬Q
T	T	F	F	F
F	T	T	F	T
T	F	F	T	T

F	F	T	T	T

Solution 2:

P	¬P	¬P^P
T	F	F
F	T	F

Solution 3:

Where hypothesis is P is $9n + 8$ is odd and conclusion Q is n is odd. But, according to the method we have to follow ¬Q→¬P

So, ¬Q is n is even then $n = 2m$ for m belongs to Integer

$9n + 8 = 9(2m) + 8$ (by substituting $n = 2m$ in $5n + 4$)

$9n + 8 = 18m + 8$

$9n + 8 = 2(9m + 4)$

$=2l$ (where $l = 9m+4$)

So, 2l is even

So, n is even then $9n + 8$ is even, so by adopting the method of Indirect proof we have hence proved the statement

Solution 4:

The different methods of proof are as follows:

1) Proof by Direct Proof Method
2) Proof by contradiction method
3) Proof by contrapositive Method
4) Proof by Induction Method
5) Proof by exhaustion
6) Constructive proof method

Solution 5:

Yes, Contrapositive is method of Proof.

1.9 SELF-ASSESSMENT QUESTIONS

1. Discuss the relationship among necessary and sufficient conditions?
2. Construct a truth table for not (A and B)
3. Construct a truth table for not (A or B)

4. Explain the difference between the method of Proof by contradiction and by contrapositive.
5. What is proof and what are different methods to prove a statement.
6. Prove that $4^n > n$ for all positive integers n by the Principle of Mathematical Induction.
7. Prove " if n is an integer, $n^2 + 3n + 2$ is even", prove by method of exhaustion.
8. Prove: If x is odd, x^2 is odd by using the method of direct proof.

1.10 REFERENCES

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LESSON 2

SET AND SET OPERATION

STRUCTURE

2.1 LEARNING OBJEVTIVES

After reading this lesson, students will be able to:

1. Understand the concept of sets and its types

2. Learn the method to represent the set
3. Understand subset, proper set and power set
4. Operation on sets
5. Venn diagram and
6. Laws of Operation

2.2 INTRODUCTION

This unit is important in the sense that it will be useful for mathematical analysis and help you understand the subsequent units in a better way. In this chapter you will understand the concept of sets, ways to represent such types of set, subset, power set, set operation such as union, intersection, difference and the properties of set.

2.3 CONCEPT OF SET

It is a collection of a well-defined collection of distinct objects that can be denoted by capital letters. It can be anything like set of vowels, set of odd numbers, and set of integers. In daily life, we confine together objects of same kinds as set of airlines companies in India; set of goods produced by a particular firm.

A set is written in a curly bracket for example,

A: set of even numbers {2, 4 , 6 , 8 ,….}, B: set of vowels { a , e , i , o , u }.

An element of set is an object which is a member of a particular set, or we can state some specific property of a set, if an item has that property, then it will be termed as an element of set. As we have seen above in a Set B where a, e, i, o, u are the elements of a set of vowels, we represent them in a curly bracket and separated the elements by a comma. Some more examples we can look at, set of prime number: {2, 3, 5, 7 …} here 2, 3 , 5 , 7 are the elements of a set of prime number.

2.3.1 Cardinality of a Set

The number of elements in a set is called cardinality of set. If X is a set, then its cardinality is denoted by |X|. For example, if X is the set of days in a week, then its cardinality is 7. Y is a set of consonants in English Alphabet then its cardinality is 21.

Set membership

Sets are generally represented by a capital letter i.e., X, Y , Z whereas small case letters are used to represent the elements of set i.e. *x* , *y*, *z*. If 'P' is a set and '*p*' is an element of this set

P, then we can write, p ∈ P where '∈' implies 'belongs to'. So, p ∈ P ⇒ p belongs to set P. If it is written as q ∉ P, it means q does not belong to set P.

2.3.2 Methods of Specifying A Set

- **ROSTER OR TABULAR METHOD**: Under this method all the elements of a set are listed in a curly bracket, as we discussed above. Here, order of an element does not matter. For example: set of even number in rolling some dice is {2, 4, 6}. It does not matter whether we write elements as {2, 4, 6} or {6, 2, 4}, both represents the same set.
- **DEFINING PROPERTY OR SET BUILDER METHOD**: Under this method we write the condition or rule which determined what elements to be part of the set. We use this method, because sometime numbers of elements in a set are infinite and to write all elements is not possible. So, it would be better to write the properties that determined the elements of a set.

NOTE: Not just infinite set are represented this way, finite set can also be represented.

For example, A: {*y*| *y* is a positive even integer less than 10}. It is read as "A is a set of those elements *y* such that *y* is a positive even integer less than 10, *y* denotes any one elements of the set A.

B: {1, 4, 9 …...,100} = ($x|x = n^2$ where n is natural number less than equal to 10}

N: {1, 2, 3, 4 ……...} = Set of all-natural number

W: {0, 1, 2, 3 …….} = Set of whole number

2.3.3 Types of Set

There are two types of sets:

- FINITE SET: In this set, we can count the elements of a set i.e., they are finite or limited. As we have seen the vowels, it is a finite set as we can count the number of elements in a set.
  A = {2, 4, 6}

  B = {5, 10, 15 ….., 5000}
- INFINITE SET: In this set the elements of a set are infinite i.e., they are unlimited, counting goes on and never end. We use dot symbol (…….) that means "so on" to

represent that counting are going and never end. For example, set of prime number {2, 3, 5, 7……}, as counting goes on and it never end. It is also not possible to write all the elements of a set. So, we represent this infinite set using dots (…...). Such set commonly used in economics.

W: Set of whole number; {0, 1, 2, 3 ……...}

Y: {*y*| *y* is a line parallel to *y* axis}

C: {*x* | *x* is a line passing through a fixed point}

NOTE:

- In a set, order of an elements does not matter i.e. whether we write set of even number from 1 to 10 as {2, 4 , 6 , 8 , 10 } or { 4 , 2 , 8 , 10 , 6 } it does not change anything.
- Set: {*a*, *b* , *c* } = {*a* , *a* , *b* , *c* } because repetition of some elements does not change the set.

OTHER TYPES OF SETS

EMPTY SET / NULL SET / VOID SET = A set which contains no elements is termed as an empty set or void set or null set. For example,

X: set of triangles with more than 3 sides. It is a null set because there is as such no triangle which has more than 3 sides. It is represented by Ø or {}.

A: {*x* | *x* is a point common to two parallel lines}

B: {x^2 = 16 , *x* is odd }

EQUAL SET = If all the elements in the two sets are equal then it is termed as an equal set.

- X = {*x* | *x* is a positive even number up to 10}

  Y = {*y* | *y* = 2*n* where $1 \leq n \leq 5$}

  So, X = {2, 4 , 6 , 8 , 10 } and Y = { 2 , 4 , 6 , 8 , 10 }

Here, both X and Y have the same elements, so the set is termed as equal set.

- If A = {1 , 2 , 2 , 5 , 6 } , B = { 1 , 1 , 6 , 2 , 5 }

Here, 2 is repeating in set A and 1 is repeating in set B but still the elements 1, 2 , 6 , 5 is same in both X and Y set because it is termed as an equal set.

EQUIVALENT SET = In equal set we see the two or more sets have same elements but in equivalent sets two or more sets have same cardinality.

For example, A = { 2 , 4 , 6 , 8 }   B = { 1 , 3 , 5 , 7 }

They are equivalent set because both have same cardinality i.e. 4 but they are not equal because both containing different elements.

SINGLETON SET / UNIT SET = The set which contains only one element is termed as a singleton set.

For example, $A = \{ y \mid y \in N,\ 2 < x < 4 \}$ i.e. A = {3} only one element belongs to this set because it is the only natural number between 2 and 4.

B = {Ø}; It is also a singleton set because it contains Ø as an element.

DISJOINT SET = Two set are disjoint when they have no element in common.

For example, A = {2, 4, 6} Set of even elements in rolling a dice

B = {1, 3, 5} Set of odd elements in rolling a dice

Here Set A and B are disjoint because they have no element in common.

OPEN SET = A set is said to be open when boundaries of a set is not included in the set.

For example, $A = \{ y : 0 < y < 2 , x \in \mathbb{R} \}$

Set A is an open set as the boundary point 0 and 2 are not included.

$B = \{ z > 5, z \in \mathbb{R} \}$

Set B is also an open set, as it contains all real number greater than 5 and boundary point 5 is not included in the set.

CLOSED SET = A set is closed when boundaries of set is included in the set.

For example, $Z = \{ 4 \le x \le 7 , x \in \mathbb{R} \}$

Z is a closed set as boundary point 4 and 7 both are included in the set.

UNIVERSAL SET = It is the set which contains all the elements under the consideration in a given context, without any repetition. It is usually represented by 'U'.

For example, P = { 1 , 3 , 5 , 7 , 8 } , Q = { 2 , 3 , 5 , 4 , 9 } , R = { 1 , 3 , 4 , 6 }

Here, U = { 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 }.

IN-TEXT QUESTIONS

1. Classify the following as finite or infinite set:
   (a) Set of all complex numbers
   (b) Set of dishes on a menu
   (c) Set of even numbers less than 15
   (d) Set of real numbers between 9 and 10
2. Write down the given statement in set builder form:
   'The set of rational number up to 100'.
3. Give an example of any empty set.
4. Check whether the following is empty, equal, singleton or equivalents set.
   (a) Set of natural number $8 < x < 9$
   (b) $A = \{1, 2, 3, 4\}$ $B = \{1, 2, 6, 8\}$
   (c) $X = \{a, b, c\}$ $Y = \{a, b, b, c\}$
5. Let, $Z = \{10, 20, 30\}$ and $Y = \{8, 16\}$
   (a) Check whether:
      (i) $\emptyset \in Z$
      (ii) $16 \in Y$
   (b) Write the cardinality of Z and Y.
6. What is the cardinality of null set.

2.4 SUBSET

A set X is a subset of Y if all the elements of X are in Y or we can say X is included in Y.

For example, A = Set of even natural number

B = Set of natural number

Here, $A \subseteq B$ because set of even natural number is a part of set of natural number.

2.4.1 PROPER SUBSET

A Set X is a proper subset of set Y if every element of X is in Y but there exist at least one element in Y which is not in set X. It can be written as $X \subset Y$.

For example,

Let, X = { *a* , *z* , *b* } and Y = { *a* , *z* , *p* , *d* , *b* }be two sets

$X \subset Y$ because elements of X i.e. *a* , *z* , *b* are present in Y, but $Y \not\subset X$ because element of Y i.e. *d*, *p* are not present in X. So, X is a proper subset of Y.

This can be represented by a Venn diagram which we will discuss in next section.

For example,

- A = { 1 , 5 , 4 }   B = { 4 , 8 , 12 , 1 }

  Here, $A \not\subset B$ and $B \not\subset A$ because,

  All elements of A is not in B i.e. there exist at least one element which is not in B here in this example 5 is not present in B due to which $A \not\subset B$.

  Similarly, $B \not\subset A$ because 8, 12 elements are not present in A.

- X = { *p* , *q* , *r* }   Z = { a , c , b }

  Here, X and Z are **disjoint sets** because there is no elements common between them.

2.4.2 IMPROPER SUBSET

A subset which contains all the elements present in the original set is known as improper subset.

For example, X = { 2, 3, 4} is an original set

Y = {2, 3, 4}

Y is an improper subset of X as all the elements of set X and Y are equal i.e. all the elements of set X is in set Y.

2.4.3 POWER SET

For any set A, the set of all of its subsets is known as power set of A.

Let, A = { *p* , *q* , *r* }

then, Power Set of A = P(A) = {{ p } , { q } , { r } , { p , q } , { p , r } , { q , r } , { p , q , r } , Ø }

NOTE:

- Ø i.e. null set is a subset of every set; here Ø lies in power set of A.
- If number of elements in set A is n , then number of subset = 2^n,

  Here n = 3 therefore number of subsets is 2^3 i.e. 8.

2.4.4 PROPERTIES OF SET

- Every set is a subset of itself.
- Null set { } is a subset of every set.
- If number of elements in set A = n, number of subsets = 2^n.
- If the number of elements in set A = n, number of proper subsets = 2^{n-1}.
- X = Y if and only if X ⊆ Y and Y ⊆ X.
- If X is a subset of Y, so we can say Y is a superset of X.

IN-TEXT QUESTION

7. Let, L = { 4 , 8 , 9 , 2 , 16 , 29 } and M = { 2 , 4 , 8 , 16 }

   a) Check whether M ⊂ L or L ⊂ M.

   b) Number of subsets of set M.

8. Consider, A= {6 , 10 , 0 }

   Write power set of set A.

2.5 OPERATIONS ON SETS

Like in number we do addition, subtraction, here in set we use operation such as union of set, intersection of set, difference of set, complement of set and partition of set-in details.

2.5.1 UNION OF SETS

If there are two sets A and B, then its union is equal to set which contains all the elements of either of set A or B (possibly both) but the common elements of set A and B is taken only once. It is denoted by '∪'.

Z = A ∪ B = { a | a ∈ A or a ∈ B }

For example,

X = { 2 , 3 , 9 , 16 , 15 }

Y = { 5 , 10 , 15 , 9 , 20 }

Therefore, X ∪ Y = { 2 , 3 , 9 , 10 , 15 , 16 , 20 }

In this example 9 , 15 common in both set X and Y but we take them only once.

2.5.2 INTERSECTION OF SETS

Intersection of sets means the elements which are common in those sets. So, if X is the intersection of two sets A and B it means it contains the elements which are common in both the sets.

$X = A \cap B = \{ c \mid c \in A \text{ and } c \in B \}$

For example,

- A = { *a* , *b* , *c* , *w* }　　B = { *c* , *d* , *h* , *r* }

  A ∩ B = { *c* }

- X = { 2 , 4 , 6 , 8 , 10 , 12 }　　Y = { 1 , 3 , 5 , 7 , 9 }

  X ∩ Y = Ø = {}

Here, X and Y are **disjoint set** because it contains no common elements.

The relation between union and intersection is : $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Let's take an example based on this formula,

Example: 1. In class 30 students play football, while 25 students play basketball. The numbers of students who play either football or basketball are 40. Then find the number of students who plays both football and basketball.

Solution: Let

X = students who play football

Y = students who play basketball

$n(X) = 30$

$n(Y) = 25$

$n(X \cup Y) = 40$

$n(X \cap Y)$ = students who play both football and basketball

$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$40 = 30 + 25 - n(X \cap Y)$

$n(X \cap Y) = 55 - 40$

$n(X \cap Y) = 15$ i.e. 15 students play both football and basketball.

2.5.3 DIFFERENCE OF SETS

If there are two set A and B then (A – B) i.e. difference of set refers to set of elements that belongs to A but not to B. It contains uncommon elements of set A.

$A - B = \{y \mid y \in A \text{ and } y \notin B\}$

It can also be written as A / B.

For example, A = { 1 , 3 , 6 , 5 , 8 , 12 }

B = { 2 , 4 , 6 , 7 , 8 , 9 , 5 }

A – B = { 1 , 3 , 12 }

B – A = { 2 , 4 , 7 , 9 }

2.5.4 COMPLEMENT OF SETS

It refers to all the elements of a universal set excluding the given set, it is denoted by A^C or A′.

$A' = \{x \in U \mid x \notin A\}$ i.e. difference between universal set and A.

$(A')' = A$ i.e. double complement of set A is itself.

For example, $U = \{a, c, e, g, i, j, l, n, p\}$

$A = \{a, e, i, l\}$

$A' = \{c, g, j, n, p\}$

2.5.5 DISJUNCTIVE UNION

It is the set which contains elements present in both set X and Y except the elements which are common in both. It is also known as symmetric difference of two set. It is represented by 'Δ'.

For example, X = { 4, 8, 12, 16, 20} and Y = { 6, 8, 12}

X Δ Y ={ 4, 16, 6, 20}

2.5.6 PARTITION OF SETS

A partition of a set A is a set of one or more disjoint non-empty subsets of A such that their union makes the whole set A.

For example, U = { 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 }

A = { 2 , 3 , 4 , 5 }   B = { 1 , 6 }   C = { 7 , 8 , 9 }

Here, collection of A , B , C forms the universal set therefore, they are partition of universal set.

Example: 2. Let X = {a, b , e , i } and Y = { e , f , g , h } be two sets,

Find X ∪ Y, X ∩ Y, X – Y and Y – X.

Solution: X ∪ Y = {a, *b*, *e*, *f*, *g* , *h* , *i* }

X ∩ Y = {e}

X – Y = {a, *b*, *i*}

Y – X = {f, *g* , *h* }

Example: 3. Let X be set of boys in a class 10^{th} who scored more than 80 marks.

Y be set of boys who had attendance more than 80%

So, X ∪ Y = set of boys in class 10^{th} who scored more than 80 marks or who had attendance more than 80%.

X ∩ Y = set of boys who had scored more than 80 marks and who also had attendance more than 80% in class 10^{th}.

X – Y = set of students who scored more than 80 marks but not had attendance more than 80% in class 10^{th}.

Y – X = set of students who had attendance more than 80% but not had marks more than 80 in class 10^{th}.

Example: 4. Let U be the universal set of all players in a college.

A denotes the set of volleyball players

B denotes the set of kabaddi players

C denotes the set of badminton players

D denotes the set of table tennis players

So, A ∪ B consists of set of players who either play volleyball or play kabaddi

A^C = U – A, consists of players who do not play volleyball

C ∩ D, consists of players who play badminton and table tennis

B – (C ∩ D), consists of kabaddi players who do not play badminton and table tennis

(A – B) ∪ (A – D), consists of either players who play volleyball but not play kabaddi or players who play volleyball but not play table tennis.

IN-TEXT QUESTION

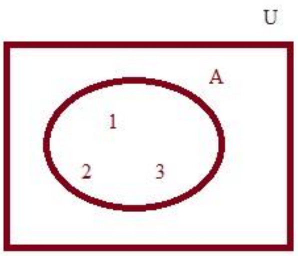
9. A = { 2 , 12 , 20 , 25 } B = { 5 , 20 , 25 , 18 }

Find A ∪ B , A ∩ B , A – B , B – A, A Δ B.

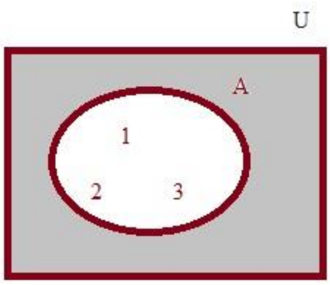
2.6 VENN DIAGRAM

It is a way to represent the elements of a particular set in a closed region of the plane. Here, the rectangle represents U i.e., universal set

- A = { 1 , 2 , 3 }

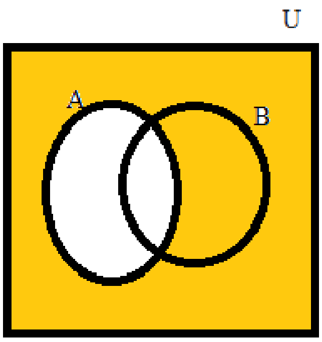


- U – A = A′



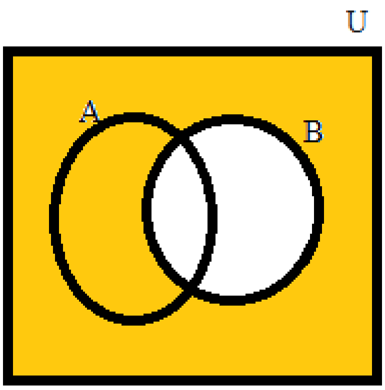
In this diagram the shaded portion is A′.

- A′



In this diagram the shaded portion is A′

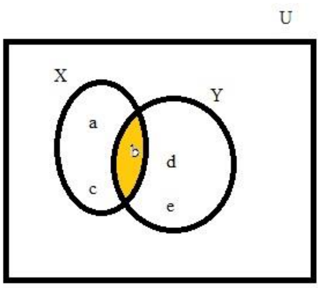
- B′



In this diagram the shaded portion is B′

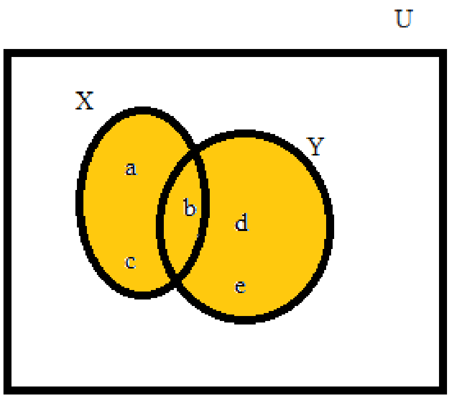
- X = {a , b , c }   Y = { b , d , e }

  Then, X ∩ Y = {b }



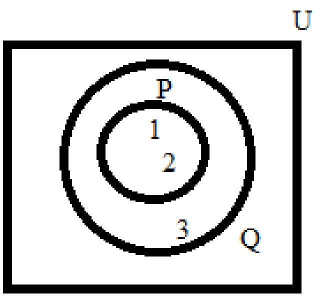
Here, shaded region is X ∩ Y

Also, X ∪ Y = {a , b , c , d , e }



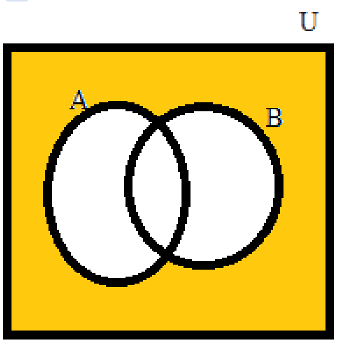
Here, shaded region is X ∪ Y

- P = {1 , 2 }   Q = { 1 , 2 , 3 }



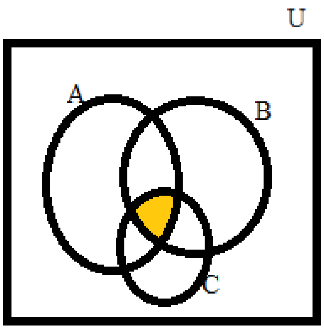
Here, P ⊂ Q

- A′ ∩ B′



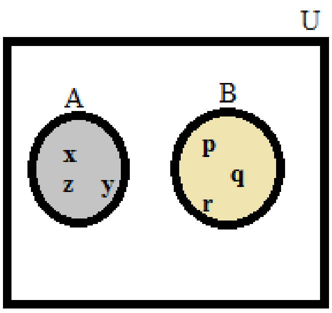
In this diagram the shaded portion is A′ ∩ B′. We can obtain the same Venn Diagram for (A ∪ B) ′

- A ∩ B ∩ C



In this diagram the shaded portion is A ∩ B ∩ C

- Disjoint sets



Here, A and B are disjoint sets because there are no elements common between them.

2.7 PROPERTIES OF SET

1. COMMUTATIVE LAW
   - A ∪ B = B ∪ A
   - A ∩ B = B ∩ A
2. ASSOCIATIVE LAW
   - X ∪ (Y ∪ Z) = (X ∪ Y) ∪ Z
   - X ∩ (Y ∩ Z) = (X ∩ Y) ∩ Z

If a parentheses position is changed, it does not change the resultant set.

3. IDEMPOTENT LAW
   - A ∪ A = A

- $A \cap A = A$

4. DISTRIBUTIVE LAW
   - $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
   - $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
5. DE-MORGAN'S LAWS
   - $(A \cup B)' = A' \cap B'$
   - $(A \cap B)' = A' \cup B'$

Example: 5. $X = \{1, 2\}$ $Y = \{2, 4\}$ $Z = \{2, 4, 6\}$

$X \cap Y = \{2\}$ $(X \cap Y) \cap Z = \{2\}$

$Y \cap Z = \{2, 4\}$ $X \cap (Y \cap Z) = \{2\}$

Therefore, associative law is verified i.e. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

Example: 6. $X = \{5, 10, 15\}$ $Y = \{10, 20\}$ $Z = \{5, 25\}$

$X \cap Y = \{10\}$ $X \cap Z = \{5\}$

$(X \cap Y) \cup (X \cap Z) = \{5, 10\}$

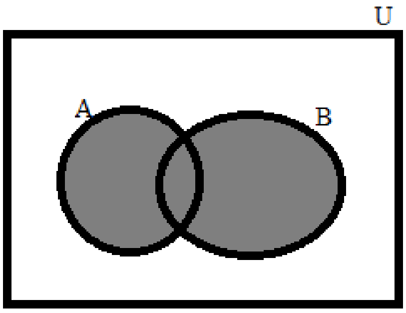
$Y \cup Z = \{5, 10, 20, 25\}$

$X \cap (Y \cup Z) = \{5, 10\}$

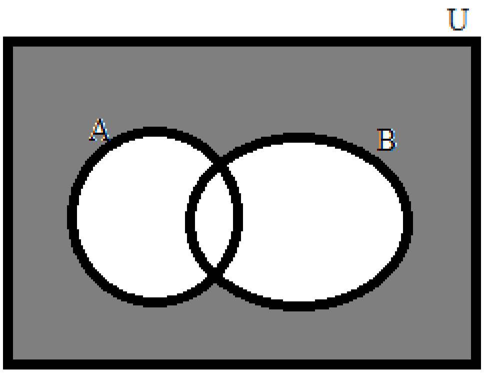
Therefore, distributive law is satisfied i.e. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

From VENN DIAGRAM we prove this De-Morgan's law

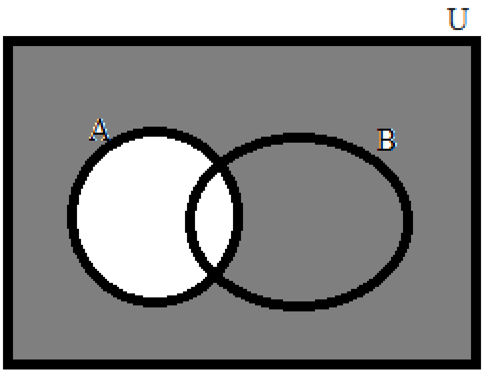
- $(A \cup B) =$



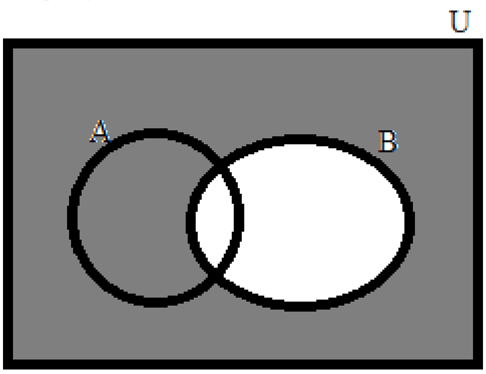
- $(A \cup B)' =$



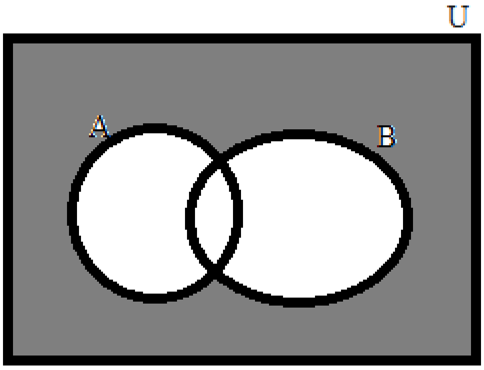
- A′



- B′



- $A' \cap B'$



From the diagram, we have proved that $(A \cup B)' = A' \cap B'$

IN-TEXT QUESTION

10. Using the Venn diagram, proof that $(A \cap B)' = A' \cup B'$

11. Check whether the following formula are true or false.

    $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$

    If false, correct the formula and prove it using the example.

2.8 SUMMARY

In this chapter we discussed about the definition of set, its types, and ways to represent set. Then we look at subsets and performed operation such as union, intersection, complement, difference of set and these will be useful in the further chapters. We also used a Venn diagram to represent the sets in a closed region and at last we performed laws of operation.

2.9 ANSWERS TO IN-TEXT QUESTIONS

1. (a) Infinite

   (b) Finite

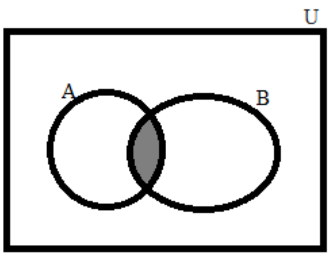
   (c) Finite

   (d) Infinite

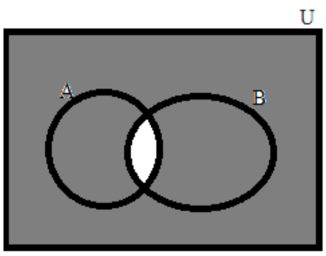
2. $A = \{ x \mid x \text{ is a rational number up to } 100\}$

3. Natural number between 3 and 4

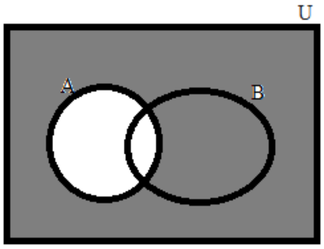
4. (a) Empty
   (b) Equivalent
   (c) Equal
5. (a) Yes, it belongs to
   (b) Yes, it belongs to
   (c) Z cardinality = 3, Y cardinality = 2
6. 0
7. (a) $M \subset L$
   (b) 2^4
8. { {0},{6}, {10}, {0,6}, {0,10}, {6,10}, {0,6,10}, Ø}
9. $A \cup B = \{2, 5, 12, 20, 25, 18\}$
   $A \cap B = \{20, 25\}$
   $A - B = \{2, 12\}$
   $B - A = \{5, 18\}$
   $A \Delta B = \{2, 12, 5, 18\}$
10.
   - $A \cap B$ =



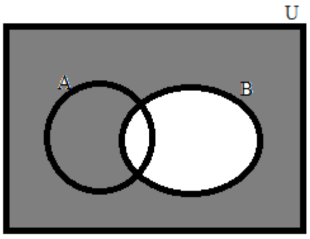
   - $(A \cap B)'$ =



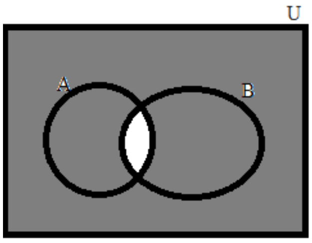
- A′ =



- B′ =



- A′∪ B′



Hence. Proved that (A ∩ B)′ = A′∪ B′

11. 11 False

2.10 SELF-ASSESSMENT QUESTIONS

1) Consider the following sets

   X = {2 , 4 , 8 , 16}   Y = {4 , 9 , 10}   Z = {4 , 9 , 15}

   and W = { 9 }

   Find X∩Y, X∩Z, Y∩Z, Y∪Z, Y – W, Z – X, X∩Y∩Z.

2) Make a list of all subsets of set {2, 0, 4}.

3) Check whether the following formulas are correct or not.

   (a) X – Y = Y – X

   (b) n(A∪ B) = n(A) + n(B) – n(A∩ B)

(c) $A \subseteq B$ implies $A \cup (B - A) = A$

4) Let X and Y are two disjoint sets then find X intersection Y.

5) In a class, 20 students drink tea, 15 drink coffee. There are 30 students who drink either tea or coffee. Find the number of students who drink both tea and coffee.

6) In a group of 50 people, 25 speak English and 40 speak Hindi. Find the number of people in a group who speak only hindi and how many people speak both.

2.11 REFERENCES

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LESSON 3

RELATION AND FUNCTION

STRUCTURE

3.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to understand:

1. the concept of ordered pairs & cartesian product
2. the concept of relation and its properties
3. functions and its types and
4. domain, co-domain & range

3.2 INTRODUCTION

In the last chapter, we learned about set. In this chapter we will be using the concepts of sets such as subset, power set and operation of sets. Here, we will start with introducing ordered pairs and cartesian products. Then we will discuss about relation, function, domain, co-domain and range. We will also define function and its types. In subsequent chapters will discuss about the different kinds of functions.

3.3 ORDERED PAIRS AND CARTESIAN PRODUCT

In the last chapter, we discussed about the set theory, in which we discussed that order in which elements are written does not matter. It is same whether we write {2 , 4 , 6 } or { 4 , 2 , 6 }. But if order matter then (2 , 4 , 6) and (4 , 2 , 6) represents two different pairs. These are called ordered sets and it is represented by parentheses ( ).

Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal. Considered an ordered pair (p , q) where p ∈ P and q ∈ Q.
Where, P = Price of commodities in rupee (1 , 2 , 3 , 4 ,.....,10)

Q = Quantity demanded of commodities in kg (10, 9 , 8 , 7,.....,1)

So, ordered pair (1 , 2) represent that at price = 1, quantity demanded = 2,

Whereas, (2 , 1) represents that at price = 2, quantity demanded = 1.

CARTESIAN PRODUCT

Given two non – empty sets A and B, the Cartesian product A× B is the set of all ordered pairs of elements from A and B i.e.

$A \times B = \{ (a , b) \mid a \in A, b \in B\}$

We read A × B as A cross B.

If A = {1 , 2 , 3 }   B = { 4 , 6 }

A × B = { 1 , 2 , 3 } × { 4 , 6 }

= { (1 , 4) , (1 , 6) , (2 , 4), (2 , 6), (3 , 4), (3 , 6) }

Here, 1st element belongs to A and 2nd element belongs to B.

IMPORTANT POINTS TO REMEMBER

- If (x , y) ∈A × B then x ∈ A and y ∈ B.
- If there are x elements in A and y elements in B, then there will be x.y elements in A × B.

  i.e. if n(A) = x *n* (B) = y

$n(A \times B) = x.\, y = n(A).\, n(B)$

- If $A \times B = \emptyset$ then either $A = \emptyset$ or $B = \emptyset$.
- $A^3 = A{\times}A{\times}A = \{ (x , y , z) : x , y , z \in A\}$

Here, (x , y , z) is called as ordered triplet.

$A^n = A \times A \times A \times ...... \times A$ here A is taken N time.

IN-TEXT QUESTION

1. $P = \{ a , d , f \}$ $Q = \{ e , h \}$

   Check whether $P \times Q$ is equal to $Q \times P$.

3.4 RELATIONS

A Relation R from a non- empty set X to a non- empty set Y is a subset of the Cartesian product X × Y.

$A = \{ a , b \}$ $B = \{ x , y , z \}$

$$A \times B = \{ (a , x) , (a , y) , (a , z) , (b , x) , (b , y) , (b , z) \}$$

$$n(A \times B) = n(A).\, n(B) = 2.3 = 6$$

Number of subsets $= 2^n = 2^6 = 64$

$R \subseteq A \times B$

As we can take $R_1 = \{ (a , x) , (b , y) : p \in A , q \in B\}$

Example: 1. If $A = \{ 1 , 2 , 4 \}$ $B = \{ 3 , 5 \}$

$$A \times B = \{ 1 , 2 , 4 \} \times \{ 3 , 5 \}$$

$$n(A \times B) = 6$$

Number of subsets $= 2^n = 2^6 = 64$

- $R_1 = \{ (x , y) \mid x \in A , y \in B$ such that $x + y$ is even $\}$

  $= \{ (1 , 3) , (1 , 5) \}$
- $R_2 = \{ (x , y) \mid x \in A , y \in B$ such that $x + y$ is an exact multiple of 4 $\}$

= { (1 , 3) }

- R_3 = { (x , y) | x ∈ A , y ∈ B such that x < y }

  = { (1 , 3) , (1 , 5) , (2 , 3) , (2 , 5) , (4 , 5) }

3.4.1 DOMAIN AND RANGE OF RELATIONS

- Domain of the relation: It is the set of all first element of the ordered pairs in relation R from set P to set Q.
- Range of the relation: It is the set of all second elements of the ordered pairs in relation R from set P to set Q.

Example: 2. A = { 2 , 3 , 4 } B = { 1 , 5 , 8 , 4 }

Let R be a relation 'is less than ' from A to B. What is Domain and Range of R.

Solution: R = { (2 , 5) , (2 , 8) , (2 , 4) , (3 , 4) , (3 , 5) , (3 , 8) , (4 , 5), (4 , 8) }

Domain: {2 , 3 , 4 }

Range : { 4 , 5 , 8 }

3.4.2 TYPES OF RELATIONS

1. EMPTY RELATION: When no element of P is related to any element of P

   i.e. R = Ø ⊂ P × P.

2. UNIVERSAL RELATION: When each element of P is related to every element of P i.e. R = P × P.

3. INVERSE RELATION: Let R be a relation from P to Q. Inverse relation (R^{-1}) is a relation from Q to P.

R = { (1 , 2) , (2 , 4) , (3 , 6) }

R^{-1} = { (2 , 1) , (4 , 2) , (6 , 3) }

3.4.3 PROPERTIES OF RELATIONS

A relation R in a set P is called

1. REFLEXIVE: If for every p ∈ P, (p , p) ∈ R.
2. SYMMETRIC: For every p_1 , p_2 ∈ P , (p_1 , p_2) ∈ R implies that (p_2 , p_1) ∈ R.

3. TRANSITIVE: If $(p_1, p_2) \in R$ and $(p_2, p_3) \in R$ implies that $(p_1, p_3) \in R$ for all $p_1, p_2, p_3 \in R$.

*** EQUIVALENCE RELATION:** R is an equivalence relation when it is reflexive, symmetric and transitive.

Example: 3. Let P = { a , b , c }

R = { (a , b) , (b , c) , (a , a) , (b , b) }

Is this relation equivalence?

Solution: For relation to be equivalence it has to be reflexive, symmetric and transitive.

It is not reflexive because (c , c) does not belong to a set.

It is not symmetric as (b , a) , (c , b) does not belong to a set.

It is not transitive as (a , c) does not belong to a set because as (a, b) , (b , c) then (a , c) need to belong to R for relation to be transitive.

Therefore, this relation is not an equivalence relation.

IN-TEXT QUESTION

2. For B × A, find relation such that x + y is prime number.

   A = { 1 , 2 , 4 } and B = { 3 , 5 }

3. X = { 1 , 2 , 4 } Y = { 2 , 4 , 6 , 8 }

   R be a relation " is 2 less than" from A to B.

   Find Domain and Range.

4. R = { (x , y) : x , y ∈ N and $x^2 = y$ } , then what is the relation.

   (a) Symmetric

   (b) Reflexive

   (c) Transitive

   (d) None of these

3.5 FUNCTIONS

When two or more different ordered pairs do not have the same first element then R is said to be a function.

Function: $y = f(x)$

i.e. for each value of x , unique value of y exists.

Let's understand with the help of following relation.

- $R = \{ (1, 3), (3, 7), (5, 11) \}$

  It is a function because in this ordered pair first element is not same.

- $R = \{ (2, 4), (1, 6), (1, 7) \}$

  It is not a function because first element same in ordered pair (1 , 6) , (1 , 7) but second element is different.

- $R = \{ (3, 5), (6, 0), (3, 5) \}$

  It is a function because in this if first element same in two ordered pairs, then second element also same i.e. same ordered pair.

- $R = \{ (1, 6), (2, 4), (-2, 4) \}$

  It is a function because even though second element is same in two ordered pairs (2 , 4) , (-2 , 4) but first element is different.

Example: 4. $X = \{ 1, 6, 4 \}$ $Y = \{ 3, 5, 8 \}$

For relation, $R = \{ (6, 8), (4, 3), (4, 5) \}$

Here for x = 4 , y = 3 and for the other elements x = 4 , y = 5 i.e. for single value of x = 4 , two value of y is associated i.e. 3 and 5.

Therefore, it is not a function.

3.5.1 DOMAIN , CO-DOMAIN and RANGE DOMAIN

Let $y = f(x)$ or $f : X \rightarrow Y$

Value of X at which f (X) is well defined i.e. always defined.

We will discuss the domain of different kinds of functions.

- CASE : 1 POLYNOMIAL FUNCTION

For example, $y = x^2 + 1$

$y = 2x + 1$

$y = 5x^0$

here, in all cases domain is $\mathbb{R}$ i.e. $x \in \mathbb{R}$.

For any real value of X , function is well defined.

- CASE : 2 RATIONAL FUNCTION

It is a function in p/q form, where $q \neq 0$.

For example, $y = (x+2) / (x-3)$

Here, $x - 3 \neq 0$

i.e. $x \neq 3$

$x \in \mathbb{R} - \{ 3 \}$

Function is well defined for any real value of x except 3.

Consider $y = (2x - 1) / x$

Here, $x \in \mathbb{R} - \{ 0 \}$, for function to be well defined because denominator cannot be equal to 0.

- CASE: 3 EVEN NUMBER OF ROOTS

It is in the form of $\sqrt{f(x)}$, $\sqrt[4]{f(x)}$, $\{f(x)\}^{-1/2n}$

Here, function is well defined when $f(x) \geq 0$ i.e. under root value is positive.

For example: $y = \sqrt{(2x + 4)}$

$2x + 4 \geq 0$

$x \geq -2$

i.e. $x \in [-2, \infty)$

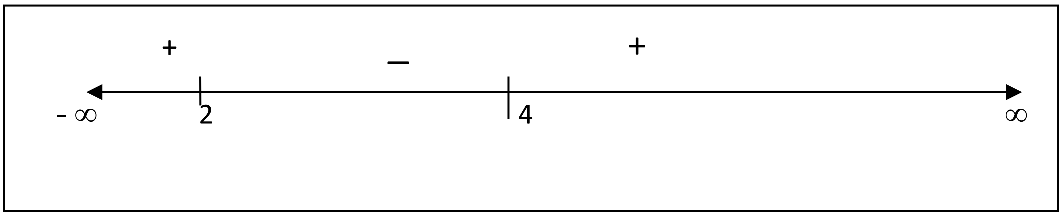
Let us consider another example,

$y = \sqrt{(x^2 - 6x + 8)}$

function is well defined when $x^2 - 6x + 8 \geq 0$

$x^2 - 4x - 2x + 8 \geq 0$

$(x - 4).(x - 2) \geq 0$



We draw a number line to check the positive value of this term.

Therefore, $(-\infty, 2] \cup [4, \infty)$ in this interval function is well defined.

See, here we introduced union symbol that we discussed in last chapter i.e. set theory.

- CASE: 4 LOGARITHM FUNCTION

Here if $y = f(x) = \log x$

It is well defined when $x > 0$

For example: $y = \ln(\ln x)$ $[\ln x = \log_e x]$

Function is well defined when

$\ln x > 0$ i.e. $\log_e x > 0$

$x > e^0$

$x > 1$

Domain: $(1, \infty)$

$$\begin{bmatrix} \log_a x = y \\ x = a^y \end{bmatrix}$$

RANGE:

The set of all resulting values of $y = f(x)$ given $x \in X$.

CASE: 1 If domain $\in \mathbb{R}$ or $\mathbb{R} - \{\text{finite value}\}$ then we calculate first $x = g(y)$ i.e. x in terms of y. So, range would be the value of y at which $g(y)$ is well defined.

CASE: 2 If domain $\in$ some interval then range would be the value of y at that interval and when $f'(x) = 0$.

For example, $y = (x + 5)/(x - 3)$

Here, domain is where denominator is not equal to zero.

i.e. $x - 3 \neq 0$

$x \neq 3$

Domain: $x \in \mathbb{R} - \{3\}$

Range : Here, case 1 applies where domain belongs to $\mathbb{R} - \{$ Finite Value $\}$

i.e. $\mathbb{R} - \{3\}$

So , we need to calculate g (y) = x

$y = (x + 5)/(x - 3)$

$xy - 3y = x + 5$

$x(y - 1) = 5 + 3y$

$x = (5 + 3y)/(y - 1)$

here, $y - 1 \neq 0$

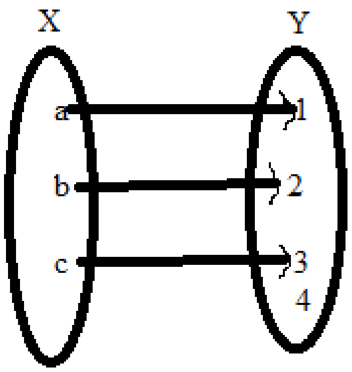
$y \neq 1$

therefore, range: $y \in \mathbb{R} - \{1\}$

CO-DOMAIN:

The Co-domain is the set of all possible value of a function. The range, which is ouputs of a function. So, range is a subset of co-domain. Will discuss more concepts related to this later in the chapter.

Consider a function: $X \rightarrow Y$



Here, domain is set X

co-domain is set Y

while possible value of Y is range is { 1 , 2 , 3 }

In this case range ≠ co-domain

IMAGE: y = f (x)

Here, y is the image of X at f

whereas x is the pre- image of Y at f.

IN-TEXT QUESTION

5. Check whether the following relation is a function or not:

   A = [1 , 3 , 5 } B = { 4 , 3 , 2 }

   (a) R = x + y is odd

   (b) R = product of x and y is odd.

6. Find domain of the following:

   (a) $y = \ln\left(\frac{2x-1}{1-x}\right)$

   (b) y = √ {(x + 1) / [(x – 2). (x – 4)] }

   (c) y = √ (2x + 3)

7. Range is always equal to co-domain.

Check whether the above statement is true or false and give explanation to your answer.

3.5.2 Types of Functions

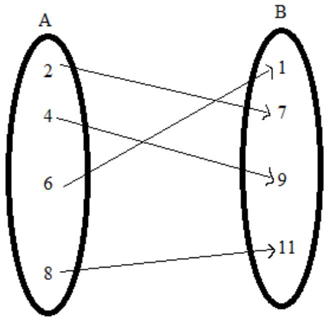
Let A , B be two non-empty sets.

f : A → B (A mapping B)

1. INJECTIVE FUNCTION: If all the elements of A has a unique image in B, then f is injective or one to one.

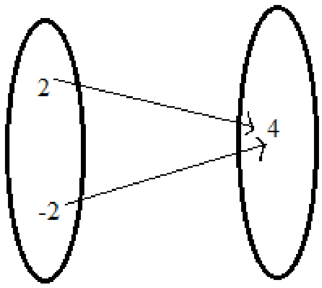
i.e. If, f (p) = f (q) implies , p = q

f (p) ≠ f (q) implies , p ≠ q



Here, $p \neq q$ i.e. $f(p) \neq f(q)$. It means for all elements of A there exists unique element in B.

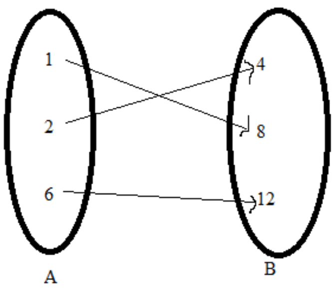
Example: Let, $y = x^2$



It is not a injective function, as for two different values in domain, there exists single image for both i.e. f(2) = f(-2) = 4. Such kind of function is known as many to one function.

NOTE: when function is strictly increasing or strictly decreasing then function is injective or one to one.

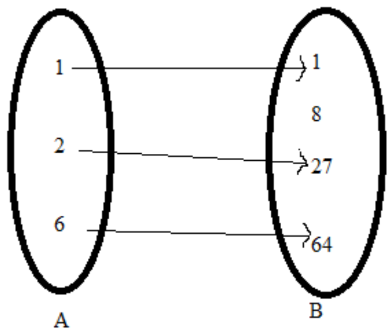
2. SURJECTIVE FUNCTION: Function is surjective if $\forall$ b $\in$ B, there exists a $\in$ A such that f (a) = b. In other words, when all elements of B has at least one pre-image in A then f is surjective or onto. In this case range is same as co-domain.



Here, range = codomain

Therefore, onto function

NOTE: A function is said to be an into function when range is not equal to co-domain. In this case, range is a subset of co-domain.

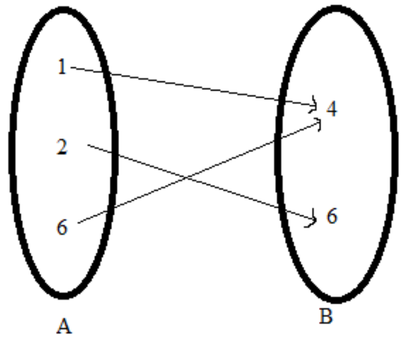


Here for 8 (i.e., element of B) there is no pre-image.

Range ≠ Codomain

Therefore, it is not a surjective function.

3. BIJECTIVE FUNCTION: The function which is both one to one and surjective then it is a bijective function. If f (a) = b, for every b in B there exists only one a in A.



It is not a bijective function because it is not one to one function.

IMPORTANT POINTS TO REMEMBER

- y = f (x) ; y is a function of x.

But we can also write x as a function of y i.e. $x = f^{-1}(y)$. But this can only be written when function is one to one correspondence, but if it is not the case then it will not follow the definition of function.

- If function is onto and one to one, then the domain of the inverse function will be Y.
- If function is into and one to one, and if we take care of domain then we can still define inverse function.

3.5.3 COMPOSITE MAPPING

If $f : A \rightarrow B$ and $g : B \rightarrow C$, we can define

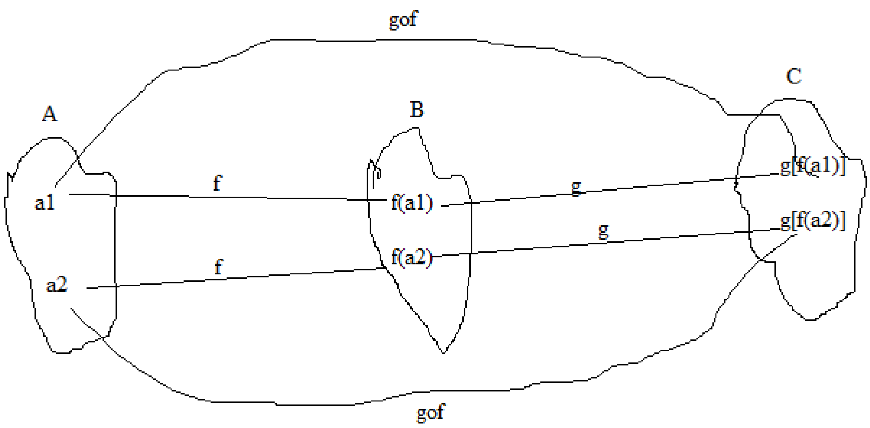
$gof : A \rightarrow C$ such that,

$c = g (f (a))$

We put $a \in A$ into f and image 'b' where $b = f(a)$, we get into g to get element $c \in C$. In this way we get a mapping from A to C.

NOTE:

- ➢ fog exists iff range of g $\subset$ domain of f.
- ➢ gof exists iff range of f $\subset$ domain of g.



Example: 5. If $f(a) = (a + \sqrt{2}) / (1 - \sqrt{2}a)$, hen what is the value of f(f(a))?

Ans: $f(a) = (a + \sqrt{2}) / (1 - \sqrt{2}a)$

$f(f(a)) = [f(a) + \sqrt{2}] / [1 - \sqrt{2}f(a)]$

$$= \frac{\left[\dfrac{a + \sqrt{2}}{1 - \sqrt{2}a}\right] + \sqrt{2}}{1 - \sqrt{2}\left[\dfrac{a + \sqrt{2}}{1 - \sqrt{2}a}\right]}$$

$$= \frac{a + \sqrt{2} + \sqrt{2} - 2a}{1 - \sqrt{2}a - \sqrt{2}a - 2}$$

$= (-a + 2\sqrt{2}) / (-1 - 2\sqrt{2}a)$

$= (a - 2\sqrt{2}) / (1 + 2\sqrt{2}a)$

Example: 6. Let f: ℕ. → ℕ

f (a) = a^2 + a +1 then prove that the f is one to one but not onto.

Ans: f (a) = a^2 + a + 1

Here, domain is set of natural number and co-domain is set of natural number. For every element in domain, there exists a unique element in co-domain. For example, if a = 1, its image is 3, a =2 its image is 7. So, for every element in domain there exist a unique image therefore, function is one to one.

For onto:

Co-domain = ℕ

Range starts from 3 (as if you put a = 1 i.e., smallest natural number, then f(a) = 3)

Co-domain starts from 1

Therefore, co-domain ≠ range

Function is not onto.

ALTERNATIVE METHOD: To check whether function is one to one we can check whether functions is increasing or decreasing. And for that we differentiate. We discuss this in detail in later chapters.

If f' (a) >0 it is increasing function

Otherwise, if f' (a) < 0 it is decreasing function.

Here, f'(a) = 2a + 1

Domain given is 'ℕ' then it will always get positive value. Therefore, function is increasing.

and we know that when function is strictly increasing then it is one to one.

Hence, proved.

Example: 7. Consider, $f : \{ p_1, p_2, p_3 \} \rightarrow \{ q_1, q_2, q_3, q_4 \}$ and

$g : \{ q_1, q_2, q_3, q_4 \} \rightarrow \{ r_1, r_2, r_3 \}$ s.t.

$f(p_1) = q_1$, $f(p_2) = q_2$, $f(p_3) = q_3$

$g(q_1) = r_1$, $g(q_2) = r_2$, $g(q_3) = r_3$, $g(q_4) = r_3$

then show that gof is bijective.

Solution: Function is bijective when it is one to one and onto.

gof = g(f(x))

- $gof = g(f(p_1))$
  $= g(q_1)$
  $= r_1$
- $gof = g(f(p_2))$
  $= g(q_2)$
  $= r_2$
- $gof = g(f(p_3))$
  $= g(q_3)$
  $= r_3$

For every element there is a unique image therefore, function is one to one.

Range = Co-domain

Therefore, function is onto.

Function is bijective.

3.5.4 INVERSE FUNCTION

Function 'h' and 'f' are inverse function when f(a) = b and h(b) = a.

For example, $f(a) = \sqrt{(-1 + a)} = b$

Then inverse, $g(b) = -1 + a = b^2$

$f^{-1} = g(b) = a = b^2 + 1$; inverse off(a)

Example: 8. If f(x) = y = 3x +4. Then find inverse off(x).

Solution: f(x) = y = 3x +4

y – 4 = 3x

g(y) = x = (y – 4)/ 3

g(y) is the inverse of f(x).

IMPORTANT POINT TO REMEMBER

INCREASING AND DECREASING FUNCTION

As we said if function is increasing or decreasing, function is said to be one to one or injective.

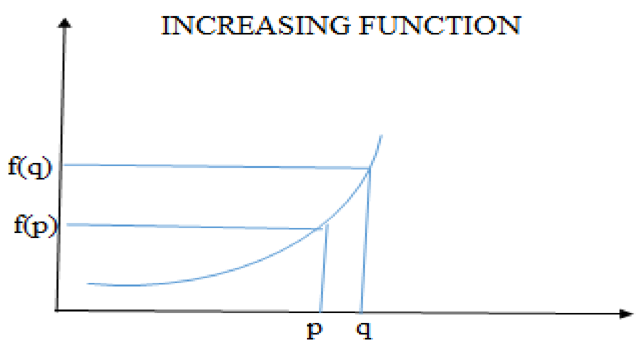
Let's discuss what is increasing or decreasing function:

- If $p < q$ then $f(p) \leq f(q)$

  This is the property of increasing function.

- If $p < q$ then $f(p) < f(q)$

  This is the property of strictly increasing function.

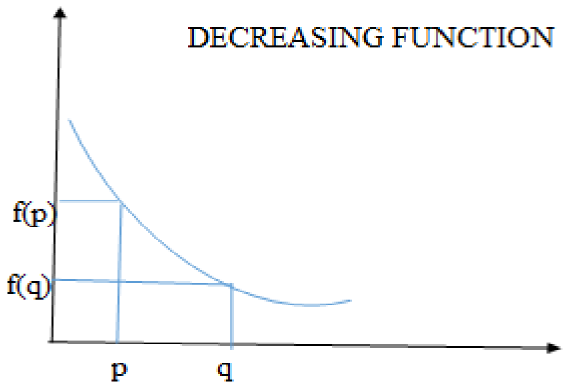


- If $p < q$ then $f(p) \geq f(q)$

  This is the property of decreasing function.

- If $p < q$ then $f(p) > f(q)$

  This is the property of strictly decreasing function.



IN-TEXT QUESTION

8. If $f: \mathbb{N} \rightarrow \mathbb{N}$ then $f(x) = 2x + 1$ is what kind of function.

   a) Bijective

   b) Surjective

   c) Injective

   d) None of these

9. $f : R \rightarrow R$

   $g : R \rightarrow R$

   $f(x) = 2x - 1$

   $g(x) = x^3 + 6$

   then $fog^{-1}(x)$ is

   a) $[(x + 11) / 2]^{1/3}$

   b) $[(x-11)/2]^{1/3}$

c) $(x/2 – 11)^{1/3}$

d) $(x – 11/2)^{1/3}$

3.6 SUMMARY

In this chapter, we have extended the concept of previous chapter. Here, we discuss about the pairs where order matters, Cartesian products of sets. Then we discussed that relation is a subset of a Cartesian product. In function we have seen the domain, range, co-domain in different kinds of functions, how equality between co-domain and range determines whether the function is onto or into.

3.7 ANSWERS TO IN-TEXT QUESTIONS

1. No, they represent different ordered pairs.

   P × Q = {(a, e), (a, h), (d, e) , (d , h) , (f , e), (f , h) }

   Q × P = {(e, a), (e, d) , (e , f) , (h , a) , (h , d) , (h , f) }

   $(a , e) \neq (e , a)$

2. R = { (3 , 2) , (3 , 4) , (5 , 2) }

3. R = { (2 , 4) , (4 , 6)}

   DOMAIN: { 2 , 4}

   RANGE : { 4 , 6 }

4. (d) none of these

5. (a) not a function.

   (b) yes, it is a function.

6. (a) (1/2, 1)

   (b) $[-1, 2) \cup (4 , \infty)$

   (c) $[-3/2, \infty)$

7. False

8. (b) injective

9. (b) $[(x-11)/2]^{1/3}$

3.8 SELF-ASSESSMENT QUESTIONS

1. Check whether relation R is set of real number

   R = { (p , q) : p > q} is reflexive, symmetric and transitive .

2. Find Range of the following function:

   $f(a) = a^2/ (1 + a^2)$

3. Let f : R → R

   f(a) = 3a + 4 then what is the inverse function.

4. Let f : N → N be defined as f(x) = 4x +1. What kind of function it is?

3.9 REFERENCES

- Hoy, M., Livernois, J., McKenna, C., Rees, R., Stengos, T. (2001). *Mathematics for Economics*, Prentics-Hall India.

LESSON 4

GRAPHS

STRUCTURE

4.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to understand:

1. Coordinates planes
2. Distance between the two pints M(x1, y1) and N(x2,y2).

3. Straight lines hold an important position in economics and determining the equation of straight line is essential, so different methods to determine the equation of straight line have been discussed.
4. Equation of circle and its derivation.
5. Graphs are an indispensable part of economics, so it is very important to learn the shifting of graphs in great detail and
6. Inequalities have been discussed; Inequality with one and two variables, linear and non-linear and absolute inequality and properties of inequalities have been discussed.

4.2 INTRODUCTION

As in the previous unit you have learned about sets, relation and function, logic and proof techniques. In this unit you will learn about elementary functions, graphs, differentiation and its application and many more. This unit is the heart of this course. To understand the language of mathematics this chapter will make you familiar with different types of graphs, equations and inequalities.

So, this chapter discusses coordinate planes, distance between two points. It also discusses about straight line and makes you familiar with the equations of straight line with its economic application i.e., equation of straight line with a point and a slope; equation of straight line with two points; equation of straight line with intercept p on y axis and with slope m.

The chapter discusses the circle and explains the derivation of the equation of circle. The graphs are an indispensable part of economics and shifting of graphs holds a crucial position in Economics. So, horizontal and vertical shifting of graphs have been discussed in great detail. The last part of the chapter discusses about inequalities and its properties it includes absolute inequality; inequality with one variable which include linear and non-linear inequality; inequality with two variables have also been discussed in the chapter.

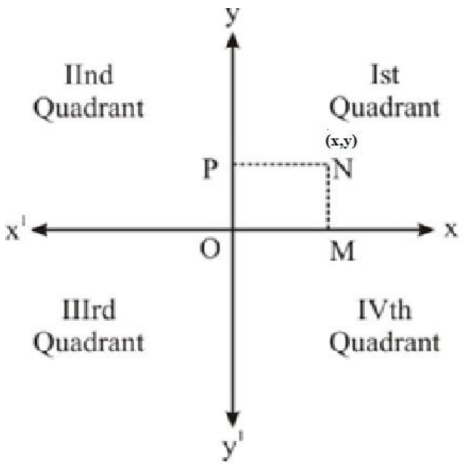
4.3 COORDINATE PLANES

A plane has two perpendicular which divides it into four quadrants. The point of intersection of two perpendicular is origin. Point N in 1st quadrant represents OM distance on x axis and OP distance on y axis

So, point N represents (x, y).

Here ray $\overrightarrow{OY}$ represent positive values,$\overrightarrow{OY}'$ represents negative values.

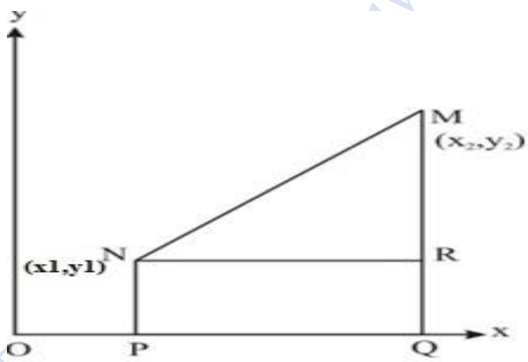
Similarly, $\overrightarrow{OX}$ represents positive values and $\overrightarrow{OX}'$ represents negative values.



Distance between two points

The distance between N and M where N is (x_1, y_1) and M (x_2, y_2).

$NR = PQ = (x_2 - x_1)$



$MR = MQ - QR = (y_2 - y_1)$

According to Pythagoras theorem in ΔMRN.

$NR^2 + MR^2 = MN^2$

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = MN^2$

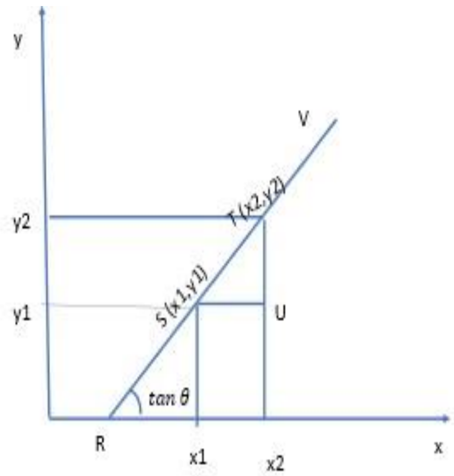
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = MN$$

So, distance between $MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

4.4 STRAIGHT LINE

Straight line passes through two points and a straight line can be represented by one point and a slope. Equation of a straight line is ax+by=c, where a,b,c are constants and x, y are variables.

4.4.1 Equation of straight line with one point and a slope equal to m.



The slope of straight line is

$$\frac{\Delta y}{\Delta x} = m = \tan\theta$$

Point T represents x and y coordinates, and slope of line RV is m.

$$m = \frac{\Delta y}{\Delta x} = \frac{TU}{SU} = \frac{y2 - y_1}{x2 - x_1}$$

as TU = $y2 - y_1$

and SU = $x2 - x_1$

So, the slope of line RV is $\frac{y2-y_1}{x2-x_1} = m$

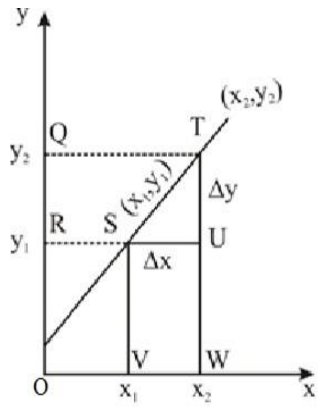
So, the equation of line $(y - y_1) = m\,(x - x_1)$ – from equation (1)

So, equation 1 is point slope form equation of straight line.

4.4.2 Equation of a straight line with two points S and T

Where S has coordinates (x_1, y_1) and T has coordinates of (x, y).

$$m = \frac{TU}{SU} = \frac{\Delta y}{\Delta x}$$



Since TU = TW – UW = $y_2 - y_1$

and SU = OW – OV

SU = $x_2 - x_1$ = WV

Since, SU = WV and TU = QR

So, slope $m = \frac{TU}{SU} = \frac{y_2-y_1}{x_2-x_1} = m$

From equation (i) we get

$$y - y_1 = m(x - x_1)$$

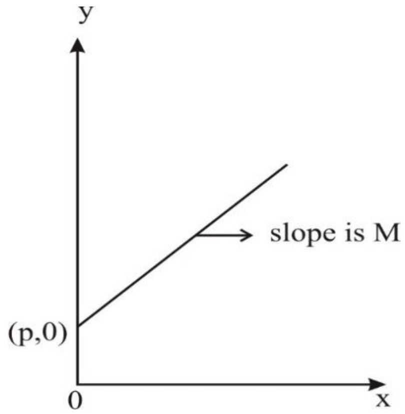
$$(y - y_1) = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x - x_1)$$

This is a two-point slope form equation of a line.

4.4.3 Equation of straight line with intercept p on y-axis with slope m.

Equation of straight line with one point and a slope as given in equation (1)

$(y - y_1) = m(x - x_1)$



(y – p) = m (x – 0)

y – p = mx

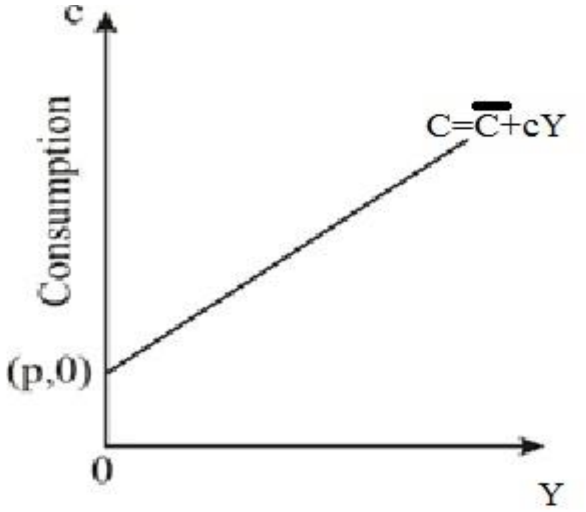
y = mx + p

This is slope intercept form of equation of line.

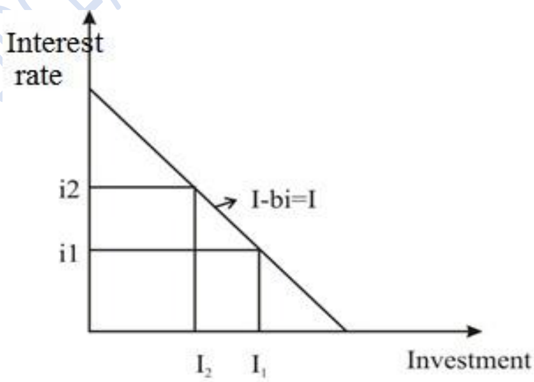
4.4.4 Use of Straight Line in Economics

Straight lines are extensively used in economics, some important functions such as consumption function, investment function and many other functions are used in economics.

Consumption function is a function of autonomous consumption and disposable income.



Similarly, investment function is a function of autonomous investment and negative function of interest rate.



Increase in the interest rate leads to decrease in investment and decrease in interest rate leads to increase in investment.

Example: A firm finds out that 30,000 units are sold when the price is Rs. 15 and 20,000 units are sold when the price is Rs. 20 per unit.

Assuming the relation between quantity demanded and price to be linear. Find the quantity demanded at Rs. 30.

Solution: The demand equation of the line passing through point (30000, 15) is represented as (x1, y1) and point (20000, 20) is represented as (x2, y2).

So, the equation of demand curve will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

We have used the equation of straight line with two points

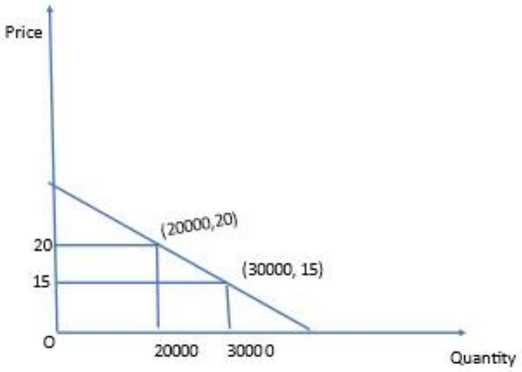
$$(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$(y - 15) = \left(\frac{20 - 15}{20000 - 30000}\right)(x - 30000)$$

$$(y - 15) = \frac{-5}{10000}(x - 30000)$$

$$y - 15 = \frac{-5x}{\boxed{10,000}} + \frac{5 \times 3\boxed{0000}}{1\boxed{0000}}$$

$$y - 15 = \frac{-x}{2000} + 15$$



$$y = \frac{-x}{2000} + 30$$

where y is price (P) and x is quantity demanded (Qd)

$$Qd = \frac{-P}{2000} + 30$$

When P=30, Qd=29.99

IN-TEXT QUESTION

1. When the price of a commodity is Rs. 20 then quantity demanded, and quantity supplied is 40 and 30 respectively. When the price of a commodity is Rs. 30 then quantity demanded, and quantity supplied is 30 and 40 respectively. Find the equations of demand and supply curve. Find the equilibrium price and quantity demanded.

2. The demand for labour in the electronic industry is Ld=1400-50W and its supply is Ls=200+50W, where L is the number of workers and W is wage rate per hour.

   i. Find the equilibrium values of L and W.

   ii. If the government wishes to increase the equilibrium wage to Rs.16 by offering a wage subsidy, find the value of L, the cost of the subsidy to the government.

4.5 CIRCLE

A circle is a locus of points with constant distance from a fixed point named as center, and fixed distance is called radius.

If the Center is at origin i.e. (0, 0) and radius (i.e., distance from the center to any point on the circle) is represented through r. So, the equation of circle is $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{r^2}$

or $x^2 + y^2 = r^2$

Similarly, the equation of circle with center as (a_1, a_2)

$(x - a_1)^2 + (y - a_2)^2 = r^2$ → Equation (1)

If we expand the given equation

$x^2 + {a_1}^2 - 2a_1x + y^2 + a_2^2 - 2a_2y = r^2$ → Equation (2)

So, we can also write the above equation

as

$x^2 + y^2 - 2a_1 x - 2a_2 y + c = 0$ → equation (3)

Where $c = a_1^2 + a_2^2 - r^2$

As we can note some features from above equation

(1) The coefficient of x^2 and y^2 are same

(2) There is no term having the product of x and y.

So, we can write equation (3) as

$kx^2 + ky^2 + 2lx + 2iy + j = 0$

Dividing equation (4) by k, we get.

$$x^2 + y^2 + \frac{2lx}{k} + \frac{2iy}{k} + \frac{j}{k} = 0$$

$$x^2 + y^2 + \frac{2lx}{k} + \frac{2iy}{k} = -\frac{i}{k}$$

Adding, $\frac{l^2}{k^2}$ and $\frac{i^2}{k^2}$ on both sides, we get

$$x^2 + \frac{2lx}{k} + \frac{l^2}{k^2} + y^2 + \frac{2iy}{k} + \frac{i^2}{k^2} = \frac{-i}{k} + \frac{i^2}{k^2} + \frac{l^2}{k^2}$$

$$\left(x + \frac{l}{k}\right)^2 + \left(y + \frac{i}{k}\right)^2 = \frac{-ik + i^2 + l^2}{k^2}$$

So, the equation of circle with center

$\left(\frac{-l}{k}, \frac{-i}{k}\right)$ and radius $\sqrt{\frac{-ik+i^2+l^2}{k^2}}$

4.6 SHIFTING OF GRAPHS

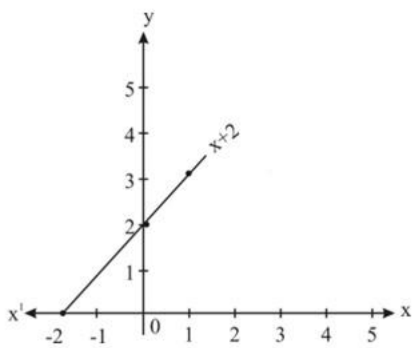
There are mainly two types of shifting. Horizontal and vertical shifts.

4.6.1 Horizontal shifting of graphs occurs when y = f(x), e is added or subtracted from x.

y = f(x + e) when e > 0 there is left ward shift in the graph.

When e < 0 there is a rightward shift in the graph.

For example, f(x) = x = y

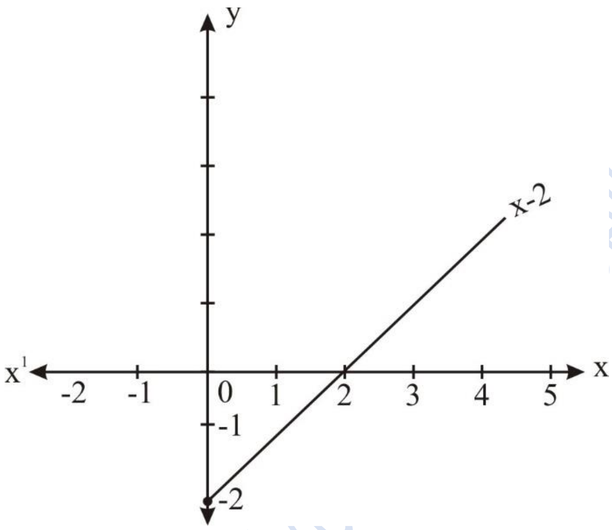


So, by adding 2 to x we get

y = f(x + 2) = x + 2

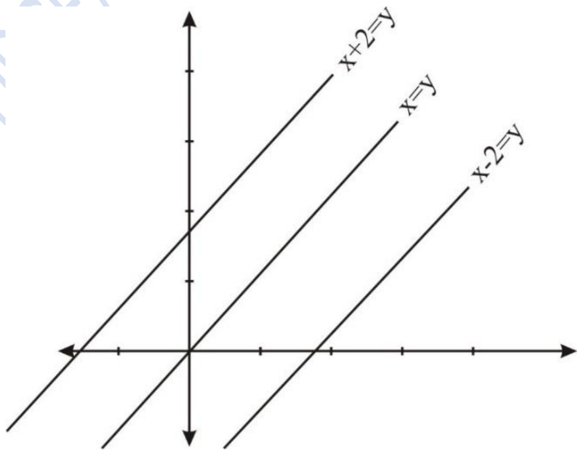
X	0	–2	1
Y	+2	0	3

by subtracting 2 for x we get



y = f (x–2) = x – 2

x	0	2	1
y	–2	0	–1



4.6.2 Vertical shifting of graph: Vertical shifting of graphs occurs when b is added to f(x).

y = f(x) + b

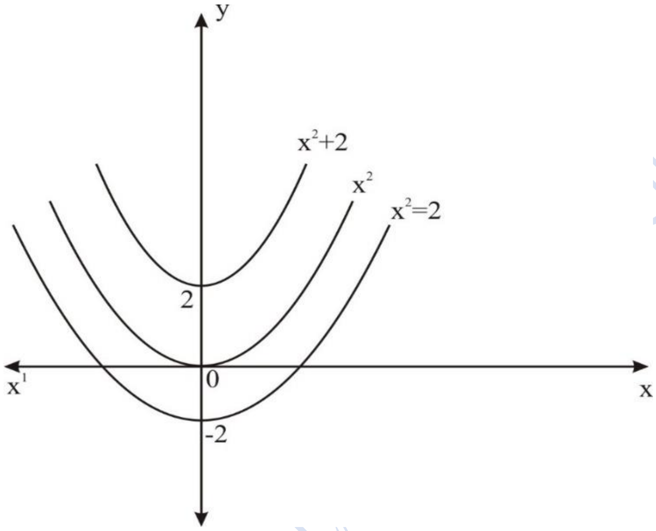
When b is positive, then graphs shift upward

When b is negative, then graphs shift downward.

for eg. $y = x^2$ and if 2 is added to f(x).

$y = x^2 + 2$ the graph will shift upward.

when $y = x^2 - 2$, the graph will shift downward.



You will study Parabola function in detail in next chapter.

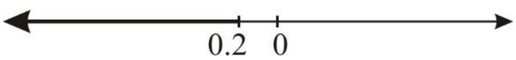
4.7 INEQUALITY

4.7.1 Inequality with One Variable

Inequality in the form of $ax + b > c$ or $ax^2 + bx + c < d$ are inequalities with one variable i.e., x.

Linear inequality

Linear inequality is in the form of $ax + b > c$.



$9x + 3 < 4x + 2$

$5x. < = 1$

$$x < \frac{-1}{5}$$

So, all the values of x less than – 0.20 are represented through thick line is the solution of the inequality.

Non-linear inequality

Non-linear inequality is in the form of $ax^2 + bx + c < d$.

$x^2 - 4x - 12 < 0$

$x^2 - 6x + 2x - 12 < 0$

$x (x - 6) + 2 (x - 6) < 0$

$(x - 6) (x + 2) < 0$

either $(x - 6) > 0$ and $(x + 2) < 0$, so, $x > 6$, $x < -2$

either $(x - 6) < 0$ and $(x + 2) > 0$, so, $x < 6$, $x > -2$

Solution is $x < 6$, $x > -2$

as $x > 6$ and $x < -2$ is not possible as the real number cannot be simultaneously greater than 6 and less than –2.

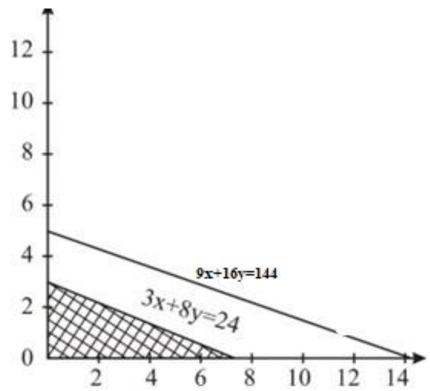
4.7.2 Linear Inequality with Two Variable

System of inequality with two variables is written as:

$3x + 8y \leq 24$

$9x + 16y \leq 144$

$x \geq 0$, $y \geq 0$



X	0	8
Y	3	0

X	0	16
Y	6	0

The area bounded by 9x + 16 y = 144 is the area below the curve 9x + 16y = 144.

Similarly, the area bounded by 3x +8y = 24 is the area below the curve 3x + 8y = 24.

So, shaded regions are the common area.

IN-TEXT QUESTIONS

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3. Sketch the area bounded by the following graph.

   $4x + 3y \leq 24$

   $6x + 8y \leq 48$

   $x \geq 0, y \geq 0$

4. solve the following inequality $\frac{2x^2+6x-8}{x+4} \leq x + 1$

4.7.3 Absolute Values

|x| is defined as follows.

$|x| = x$ if $x>0$

$|x| = -x$ if $x<0$

$|x| = 0$ if $x=0$

Example

$|x+3| \leq 2$

Solution: The inequality can be written as

$-(x+3) \leq 2$ or $(x+3) \leq 2$

$-x \leq 5$ or $x \leq -1$

$x \geq -5$ or $x \leq -1$

$-5 \leq x \leq -1$

4.7.4 Properties of Inequality

(1) If $a > b$ then $a - b > 0$ i.e., if a and b are two real numbers and if a is greater than b. So, $a - b$ is positive.

(2) If $b > a$ then $a - b < 0$ i.e., if a and b are two real numbers and if b is greater than a. So, $a - b$ is negative.

(3) Adding and subtracting any real number c on both sides will not reverse the inequality.

For eg. If $a < b$, then adding and subtracting c will not reverse the inequality.

$a \pm c < b \pm c$

(4) Multiplying a positive real number c on both the sides will not reverse the inequality.

For eg. If $a > b$ then multiplying c on both the sides will not reverse inequality.

$ac > bc$.

(5) Multiplying a negative real number c on both sides will reverse the inequality.

For eg. If $a > b$ then multiplying c on both the sides will reverse the inequality to $ac < bc$.

(6) Inequalities are transitive. i.e., if $a < b$, $b < c$ then $a < c$.

(7) If $a < b$ then $\frac{1}{b} < \frac{1}{a}$

For eg. $3 < 4$ then $\frac{1}{4} < \frac{1}{3}$

4.8 SUMMARY

A Plane is divided into 4 quadrants. And distance between any two points A as (x1,y1) and B as (x2,y2) in a plane is represented through the given formula distance AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Straight line holds a crucial position in economics. Many functions such as consumption and investment functions are linear function and are represented through a straight line. So, determination of equation of a straight line is utmost essential. The equation of straight line with a point and a slope is represented by $(y - y_1) = m (x - x_1)$. Equation of a straight line with two points is represented by $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$, where(x1,y1) and (x2,y2) are two points respectively. Equation of straight line with intercept p on y-axis with slope m is represented by y = mx + p.

If the Centre is at origin i.e. (0, 0) and radius is r then, the equation of circle is represented by $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{r^2}$.

Horizontal shifting of graphs occurs when y=f(x+e), when e>0 there is leftward shift in the graph and when e<0 there is rightward shift in the graph.

Vertical shifting in the graph occurs when b is added to the f(x) or y=f(x) +b when b is positive, then graphs shift upward when b is negative, then graphs shift downward. Linear and non-linear inequalities with one variable are of the form ax+b>c and $ax^2 + bx + c < d$ respectively. Linear inequality with two variables are used to solve the system of equations to determine the feasible region.

4.9 ANSWERS TO IN-TEXT QUESTIONS

1. The equation of quantity demanded can be found by using the formula for the equation of Straight line with two points. When the Price is Rs.20 then quantity demanded is 40. So, the Point is (20,40) Similarly the other Point is (30, 30)

So, equation of straight line with two points is $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

Where (x_1, y_1) is (20,40) and(x_2,y_2) is (30,30) So, by substituting the values in the given formula
we get,

$(y-40) = \left(\frac{30-40}{30-20}\right)(x-20)$

(y-40) = (-1) (x-20)

(y-40) = -x+20

y =- x+20+40 y=-x+60 equation 1

where y represents demand and x represents price So,

Qd = -P+60

Similarly Supply equation with (x_1, y_1) is (20, 40) and (X_2,y_2) as (30, 40) and Substituting the values in equation of straight line with two points we get

$$(y-30) = \left(\frac{40-30}{30-20}\right)(x-20)$$

y-30= 1(x-20)

y-30 = x-20

where y is Quantity supplied and x is Price
Qs = P+10 equation 2

By solving equation 1 and equation 2 we get
In equilibrium Quantity demanded = Quantity supplied
-P+60 = P+10

-2P=-50

P=25

Qd=-25+60
Qd = 35 = Qs

So, equilibrium Price is 25 and equilibrium quantity 35.

2. (i) Equilibrium number of Workers and equilibrium Wage W are determined by equating Ld=Ls.

1400 - 50w = 200+ w

1200 = 100 W

1200/100 = W 100

Rs 12 = W

Ld = 1400 - 50x12

Ld = 1400 - 600

Ls = Ld = 800

So, equilibrium wages are Rs12 and the number of workers are 800.

(ii) lf the government gives the wage subsidy to increase employment and increases the wage from 12 to 16. Then Number of labour supplied will be

LS = 200 +50w

LS = 200 +50X16

LS = 1000

and wage to be received by each labourer with 1000 labourers in the economy is

1400 - 50w = Ld

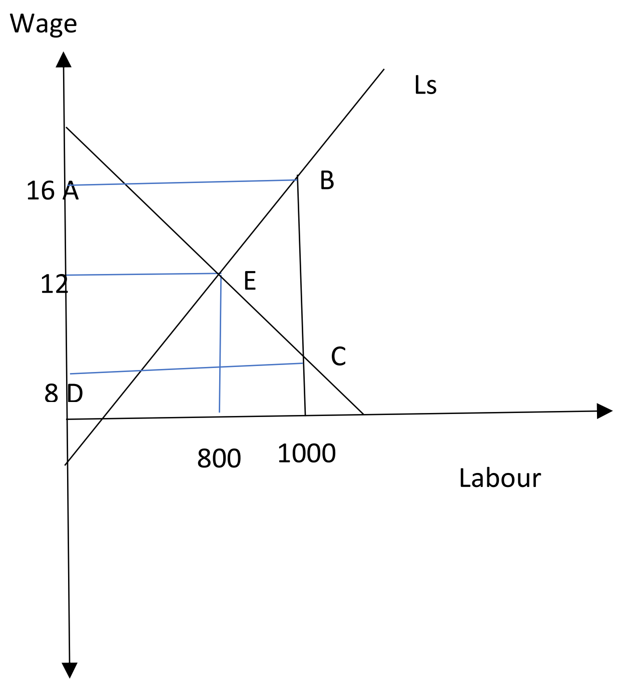
1400- 50w = 1000

400 =50w

Rs 8 = w

So, the cost of susidy to the government is the shaded region. ABCD

area ABCD = length & breadth



area ABCD = 1000 x (16-8)

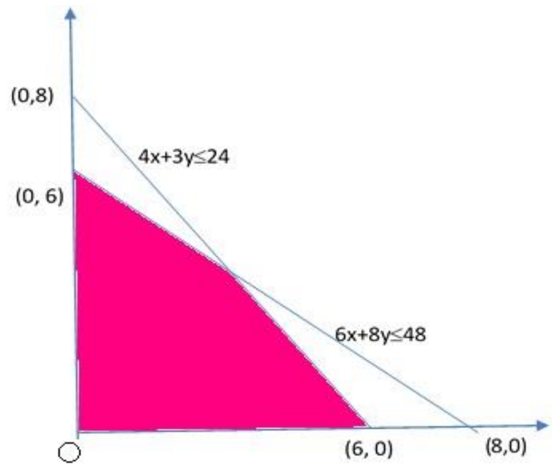
area ABCD = 1000 x 8
area ABCD = 8000

So, the cost to the government is 8000.

3. $4x+3y \leq 24$

X	6	0
Y	0	8

So, the line $4x+3y \leq 24$ passes through (6,0) and (0.8)



$6x+8y \leq 48$

X	0	8
Y	6	0

Similarly, line $6x+8y \leq 48$ passes through (0,6) and (8,0) plotting both the lines on the graph we get, the common area bounded by later inequality is shaded region

Solution 4: $\frac{2x^2+6x-8}{x+4} \leq x+1$

$$\frac{2x^2+8x-2x-8}{x+4} \leq x+1$$

$$\frac{2x(x+4)-2(x+4)}{x+4} \leq x+1$$

$$\frac{(x+4)(2x-2)}{x+4} \leq x+1$$

$$2x-2 \leq x+1.$$

$$x \leq 3$$

4.10 SELF-ASSESSMENT QUESTIONS

1. A manufacturing unit uses two factors of Production i.e., labour and capital. The Price of labour to Rs. 10 per hour and the price of capital is Rs.20 per hour. If the manufacturer wishes to spend Rs 500 per hour on Production, determine the cost equation.
2. Two Points on a linear supply relation are (45000, Rs.130) and (60000, Rs.60)
   - Find the supply equation.

- What will be in the supply when Price is Rs.80.

3. Find the Coordinates of center and radius of each of the Circles with the following equation

- $4x^2 + 4y^2 + 16x + 8y = 0$
- $2x^2 + 4y^2 - 8y = 0$.

4. Solve the in equalities

- $3 < x+2 < 5x+3$
- $2x^2 - 7x + 4 > 0$

5. Indicate the area bounded by the following system of Inequality:

- $x+y \leq 4$
- $2x-4 \geq -8$
- $x-2y \leq 8$

4.11 REFERENCES

- Sydsaeter, K., Hammond, P. (2002). *Mathematics for economics analysis*. Pearson Education.
- Hoy, M., Livernois, J., McKenna, C., Rees, R., Stengos, T, (2001). *Mathematics for Economics*, Prentice-Hall India.

LESSON 5

FUNCTIONS OF ONE REAL VARIABLES-I: POLYNOMIALS AND POWERS

STRUCTURE

5.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to:

1. Understand the quadratic functions.
2. Identify a polynomial function.
3. What is meant by 'Remainder Theorem'?
4. Evaluate the power functions.

5.2 INTRODUCTION

The linear functions that you studied in the previous chapter were too simple. Many economists find it difficult to accurately model economic phenomenon with these functions. In fact, a lot of economic models use functions that either increase or decrease until they reach a certain minimum or maximum value, respectively. In other words, when dealing with this non-linear relationship in which a change in x does not always result in a constant change in y then in that case, we use polynomials. In this chapter we will learn about the simple non-linear functions.

5.3 QUADRATIC FUNCTIONS

In simple terms, a quadratic function is a non-linear function that contains a variable that is raised to the power two (2).

For example:

Let $f(x) = ax^2 + bx + c = 0$ … (5.1)

Where, a, b and c are constants and $a \neq 0$. In the above equation if a = 0, then in that case f(x) = bx + c, then the equation becomes a linear function. As observed, variable 'x' in (5.1) is raised to power two.

Using equation (5.1), we need to find the values of x such that f(x) = 0. To begin with, we will solve it using the method known as 'completing the square'. Under this method, we will divide the whole (1) by 'a' such that:

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0 \qquad \ldots (5.2)$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x = \frac{-c}{a} \qquad \ldots (5.3)$$

Now, adding $\left(\frac{b}{2a}\right)^2$ to both the sides, we get

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 - \left[\frac{b^2 - 4ac}{4a^2}\right] = 0 \qquad \ldots (5.4)$$

Multiplying both sides by a, we get the form:

$$\Rightarrow a\left(x+\frac{b}{2a}\right)^2 - \left[\frac{b^2-4ac}{4a}\right] = 0 \qquad \ldots(5.5)$$

From (5), we find that as the value of 'x' changes, then the value of $a\left(x+\frac{b}{2a}\right)^2$ varies. If we equate this term equals to zero, then in that case $x = \frac{-b}{2a}$. If a > 0, it will never be less than zero. Given this case, if a > 0, then f(x) will attain minimum value when $x = \frac{-b}{2a}$, thus:

$$F\left(\frac{-b}{2a}\right) = \frac{-(b^2-4ac)}{4a} \Rightarrow C - \frac{b^2}{4a} \qquad (5.6)$$

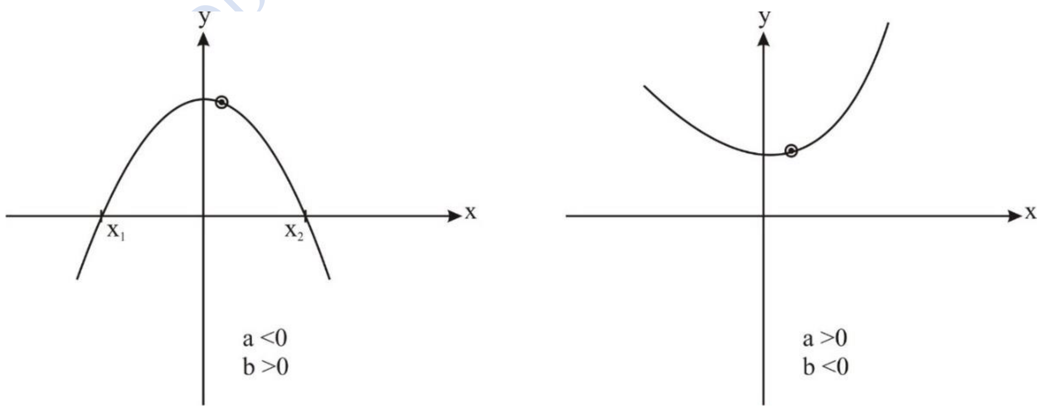
On the other hand, if a < 0 then in that case

$$a\left(x+\frac{b}{2a}\right)^2 < 0 \qquad \text{for all } x \neq -\frac{b}{2a} \qquad (5.7)$$

If $x = \frac{-b}{2a}$, then in that case, (5.7) will become zero, thus f(x) attains maximum when $x = \frac{-b}{2a}$. Thus, we conclude that:

1. $f(x) = ax^2 + bx + c$ attains minimum when a > 0, and at point $\left(\frac{-b}{2a}, C - \frac{b^2}{4a}\right)$
2. $f(x) = ax^2 + bx + c$ attains maximum when a < 0, at point $\left(\frac{-b}{2a}, C - \frac{b^2}{4a}\right)$

Graphically, the shape of the function $f(x) = ax^2 + bx + c$ will be a parabola and solutions for this equation is determined at f(x) = 0 and the intersection of this parabola with x axis. The graph will be U-shaped or inverted U shaped depending upon the values of 'a'.



The maximum and minimum points are determined as the coordinates of point A.

5.3.1 Solutions to Quadratic Equations

Example 1: Determine the roots of the following function and find maximum/minimum point of each:

(i) $f(x) = x^2 + 4x – 5$

(ii) $f(x) = –2x^2 –4x – 6$

Solution:

$x^2 + 4x – 5 = (x^2 + 4x) – 5$

here a = 1, b = 4 and c = –5, then from equation (), we will add and subtract$\left(\frac{b}{2a}\right)^2$.

$\rightarrow (x^2 + 4x + 4) – (4 + 5) \rightarrow (x + 2)^2 – 9.$

Now, this expression attains minimum value –a when x = –2.

For roots,

$(x+2)^2 = 9 \quad \Rightarrow (x+2) = \sqrt{9}$

$x+2= \pm 3$

So, x+ 2 = 3 and x + 2 = –3 $\Rightarrow \boxed{x = 1 \text{ and } x = -5}$

(b) $-2x^2 - 4x - 6 = -2(x^2 + 2x) - 6$

here a = –2, b = 2 and c = –6, then we will add and subtract $\left(\frac{b}{2a}\right)^2$

$\Rightarrow -2(x^2 + 2x + 1 - 1) - 6$

$\Rightarrow -2(x+1)^2 + 2 - 6 \Rightarrow -2(x+1)^2 - 4$

Now, the expression attains minimum value –4 when x = –1.

For determination of roots,

$–2(x+1)^2 = 4$

$(x+1)^2 = –2$

There does not exist any root as $\sqrt{-2}$ cannot be determined.

Example: Let $y = 2x^2 + 8x + 11$. Factorize the equation and draw the graph.

Solution: $y = 2x^2 + 8x + 11$ can be rewritten as

$y = 2\,(x^2 + 4x) + 11$

Using the completing square method,

$\Rightarrow y = 2\,(x^2 + 4x + 4 - 4) + 11$

$\Rightarrow y = 2\,(x+2)^2 + 3$

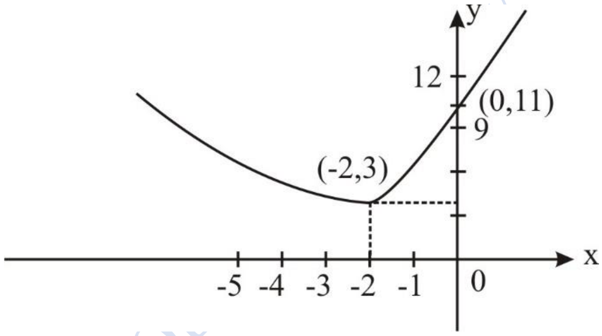
$\Rightarrow y - 3 = 2(x+2)^2$

To sketch this function, we need to find vertex such that

$y - 3 = x + 2 = 0$ i.e. $(-2, 3)$

Now, the point where the graph intersects Y axis is a point where value of x equals zero, such that

$y = 2\,(x+2)^2 + 3 = 11$ so coordinates $(0, 11)$



Example: Suppose $y = -3x^2 + 30x - 27$. Sketch the graph.

Solution: $y = +3\,(-x^2 + 10x - a) \Rightarrow -3\,(x^2 - 10x + a)$

$\Rightarrow y = -3\,(x^2 - 10x + 25 - 25 + 9)$ [$\because$ Completing the square)

$\Rightarrow y = -3[(x-5)^2 - 16] \Rightarrow -3(x-5)^2 + 48$

Again, from the previous example, vertex is determined at the points

$-3(x-5)^2 = 0$

$y - 48 = 0$

$\Rightarrow (5, 48)$

We will obtain an inverted n shaped graph in this case.

The graph will also intersect x axis, where y = 0

$$y = -3\,(x - 5)^2 + 48$$

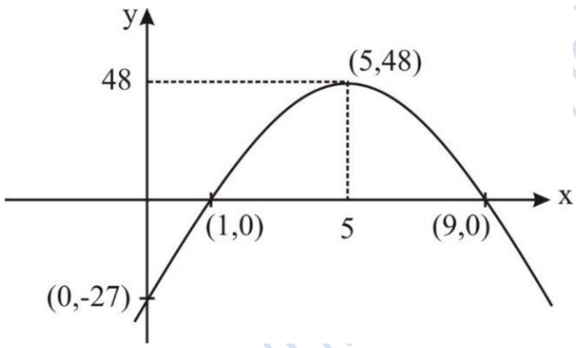
$$0 = -3\,(x - 5)^2 + 48$$

$$48 = 3\,(x - 5)^2 \Rightarrow 16 = (x - 5)^2$$

$$\Rightarrow \sqrt{16} = x - 5 \Rightarrow x - 5 = \pm 4$$

$$\Rightarrow x - 5 = -4 \quad \text{or} \quad x - 5 = 4$$

So, we get coordinates, x = 1 and x = 9, thus (1, 0), (9, 0).



The graph also intersects the y axis when x is zero at point (0, –27).

IN-TEXT QUESTIONS

1. Let $f(x) = x^2 - 7x$. Using this, complete the following table:

(a)

x	–1	0	1	2	3	4
f(x)						

(b) Determine the maximum / minimum point

(c) Find value of x for f(x) = 0

(d) Graph table in part (a) as function of 'f'.

2. Find roots of the following quadratic equations and determine the maximum/minimum points:

(a) $2x^2 + 8x + 11$

(b) $x^2 + 4x$

(c) $\frac{1}{3}x^2 + \frac{2}{3}x - \frac{8}{3}$

3. Solve the following quadratic equations:

(a) $x^4 – 8x^2 – 9 = 0$

(b) $x^6 – 9x^3 + 8 = 0$

[Hint: Put $x^2 = k$ and form quadratic equation]

4. Find solution to the following equations where ‘m’ and ‘n’ are positive parameters:

(a) $x^2 – 6mx + 4m^2 = 0$

(b) $x^2 – (m+b) x + mn = 0$

5. Find the equation of parabola that passes through three points (1, –3), (0, –6) and (3, 15).

$y = px^2 + qx + r$

[Basically, find the value of p, q and r]

ANSWERS IN-TEXT PROBLEMS

1. (a)

x	–1	0	1	2	3	4
f(x)	8	0	–6	–10	–12	–12

(b) x = 0 and x = 7

2. (a) No real solution

(b) as $b^2 – 4ac = 64 – 88 < 0$

(c) $x^2 + 4x = (x + 2)^2 – 4$ with minimum – 4 at x = –2

(d) $\frac{1}{3}(x+1)^2 - 3$, with smallest value –3 when x = –1

3. (a) x = ± 3

(b) $x^3 = 1$ or $x^3 = 8$ so x = 1 or x = 2

4. (a) x = 2m or x = 4m

(b) x = m and x = n

5. $y = 2x^2 + x - 6$, (1, –3) belongs to the graph if –3 = p + q + r

(0, –6) belongs to the graph if –6 = r

and (3, 15) belongs to the graph if 15 = ap + 3q + r

Thus, p = 2, q = 1 and r = –6

5.4 APPLICATIONS OF QUADRATIC FUNCTION

The majority of economic analysis focuses on optimization issues. As economics is associated with the study of choice, economists typically model the choice mathematically in the form of optimization problem. In this section, we demonstrate how certain fundamental economic concepts can be illustrated using the quadratic functions.

Example 1: (Profit Maximization) suppose the firm is selling Q units of goods at a price of Rs. 10 per unit, which is same at all levels of output and faces a cost curve $C(Q)=Q^2-20Q+120$. Find the level of output that maximizes profit and the corresponding level of profit.

Solution: We know that profit is computed as total revenue (P×Q) minus cost.

$\pi(Q) = (P\times Q) - \text{cost}$

$10Q - (Q^2 - 20Q + 120) \Rightarrow 30Q - Q^2 - 120$

Here

$$Q^* = \frac{-b}{2a} = \frac{-30}{2(-1)} = 15 \qquad [\because \text{Using eq.} \quad]$$

$\pi(Q) = 30(15) - (15)^2 - 120$

$\Rightarrow 450 - 225 - 120 \Rightarrow$ Rs. 105

Example 2: (Inverse demand function) suppose the firm faces an inverse demand function of the form:

$$P = \frac{231 - Q}{18}$$

and supply of the form $Q = 2P + 4P^2$

Solve for the equilibrium quantity.

Solution: We know that in equilibrium, demand equals supply

$2P + 4P^2 = 231 - 18P$ (Q = 231 – 18P is demand)

$\Rightarrow 4P^2 + 20P - 231 = 0$

Here, a = 4, b = 20, c = –231

We know that

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow P = \frac{-20 \pm \sqrt{400 + 4(231)(4)}}{2(4)}$$

$$= \frac{-20 \pm \sqrt{400 + 3696}}{8}$$

$$= \frac{-20 \pm \sqrt{4096}}{8} = \frac{-20 \pm 64}{8}$$

$$P = \frac{-20 + 64}{8} \text{ and } \frac{-20 - 64}{8}$$

P = 5.5 or –10.5

Since P can't be negative, we take P = 5.5 with equilibrium quantity as

$$Q = 2\,(5.5) + 4\,(5.5)^2 = \boxed{132}$$

Example 3: (Monopoly problem) Consider a market for vaccines. Suppose there is only one seller which is selling the Covid vaccine and thus enjoys a monopoly. The firm faces a total cost function of the form:

$$C = aQ + bQ^2 \quad \text{where } Q \neq 0$$

and a and b are positive constants. Each unit of vaccine (Q) is sold in the market at a price

$$P = \alpha - \beta Q \quad \text{where } Q \neq 0$$

and $\alpha > 0$ and $\beta \neq 0$. Find the profit-maximizing quantity and profit.

Solution: The total revenue of the monopolist is denoted as

$$TR = P.Q = (\alpha - \beta Q).\ Q \qquad \ldots (1)$$

and the profit earned by this monopolist is

$$\pi(Q) = TR - C \qquad \text{... (2)}$$

$$= (\alpha - \beta Q).\ Q - (aQ + bQ^2)$$

$$= \alpha Q - \beta Q^2 - aQ - bQ^2$$

$$= (\alpha - a)\ Q - (\beta + b)\ Q^2 \qquad \text{... (3)}$$

Since, the monopolist wants to maximize its profit levels, then he will attain maximum at the point where Q equals $\frac{\partial \pi}{\partial Q}=0$.

Using equation:

$Q^* = \frac{(\alpha - a)}{2(\beta + b)}$ and the resulting profit will be

$$\pi^* = \frac{(\alpha - a)^2}{4(\beta + b)}$$

The above equation will hold only if $\alpha > a$. If $\alpha = a$, then the monopolist will not produce any quantity and $Q^* = 0$, $\pi(Q) = 0$.

If suppose, the monopolist behaves like in a perfectly competitive market then, $\beta = 0$ and $P = \alpha$. Here the decision about price is not affected by the quantity. Putting $\beta = 0$ and $\alpha = P$ in equation (3), we get

$$\pi(Q) = (P - a)\ Q - bQ^2$$

then, $Q^* = \frac{(P - a)}{2b}$ and $\pi^* = \frac{(P - a)^2}{4b}$

with $P > a$. If $P = a$, then $Q^* = 0$ and $\pi^* = 0$.

$\Rightarrow P = a + 2bQ^*$

Now, equating this price with the demand curve of the monopolist.

$$P = \alpha - \beta Q = a + 2bQ$$

$$\alpha - \beta Q = a + 2bQ$$

$$Q = \frac{\alpha - a}{2b + \beta}$$

Thus, in this way a monopolist will behave like a perfectly competitive firm.

IN-TEXT QUESTION

1. A firm produces quantity 'Y' of a product A with total cost 3 + 2y. The demand schedule for the product $Y = \frac{1}{2}(11 - P)$, where P is the price charged for product. Determine the profit-maximizing output and the profit.

ANSWER TO IN-TEXT QUESTION

1. Demand becomes

   $P = 11 - 2Y$

   8 thus revenue is $P.Y = 11Y - 2Y^2$

$$\pi(Y) = 11Y - 2Y^2 - 3 - 2Y$$

$$= -2Y^2 + 9Y - 3$$

$$Y = \frac{9}{4}, \pi(Y) = \frac{57}{8}$$

5.5 POLYNOMIALS & CUBIC FUNCTIONS

In this section, we consider cubic functions expressed in the general form as:

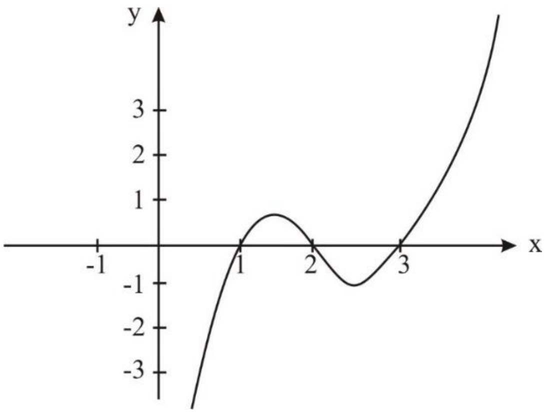
$$f(y) = ay^3 + by^2 + cy + d \quad \ldots\ldots. \quad (5.8)$$

where a, b, c and d are constants and $a \neq 0$

The solution to a cubic function is generally found by plotting the graph and finding the points of intersection on axes. The shape of the graph varies with the changes in co-efficient a, b, c and d. In most cases, the graph is generally S-shaped. Also, there will be either one root or three roots of the cubic function.

Solve the following cubic equation: $x^3 - 6x^2 + 11x - 6 = 0$

Solution: We can solve this equation by plotting the graph.



Starting with lower values of x, let < 1, x^3 also gets negative, we begin from below x line, and it moves higher to the right as x gets large and x >

0, then x^3 is increasing. The curve intersects the x axis thrice at points x = 1, x = 2 and x = 3. Thus, it implies that

$$x^3 - 6x^2 + 11x - 6 \Rightarrow (x-1)(x-2)(x-3) = 0$$

5.5.1 General Polynomial Functions

All functions such as linear, quadratic and cubic functions belong to a group of functions called 'polynomials". They can be expressed as:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots\ldots\ldots + a_1 x^1 + a_0 x^0 \quad (5.9)$$

where $a_n \neq 0$ and a_1, a_2, a_3 ……….. a_n are constant. The equation (5.9) is known as a general polynomial function of degree 'n'. If suppose, n = 5, we will get

$$f(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

that is general polynomial of degree 5.

As per the 'fundamental theorem of algebra' every polynomial expressed as (5.9) can be factored as a product of polynomials of first or second degree.

For example:

$$f(x) = x^3 - 2x^2 + x - 2$$
$$= x^2 (x-2) + 1 (x-2)$$
$$= (x-2)(x^2+1)$$

5.5.2 Integer Roots

While solving for the polynomial functions, we can get the roots of the function in the form of integer. According to the integer solution of a polynomial function, if there is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0 = 0 \qquad (5.10)$$

with co-efficient a_n, a_{n-1}, ……… a_1, a_0 as integers, then all possible integer roots of the above equation must be factor of the constant term a_0. A polynomial of n^{th} degree can have at most n roots.

Example: Find all integer roots of the equation $x^2 + x - 2 = 0$.

Solution: According to the integer theorem, all integer roots of the equation must be a factor of –2. As $x^2 + x = 2 \Rightarrow x(x+1) = 2$. Thus, it has two root x = 1 and x = –2.

IN-TEXT QUESTION

1. Find all the possible integer roots of the following equations:

   (a) $x^5 - 4x^3 - 3 = 0$

   (b) $2y^3 + 11y^2 - 7x - 6 = 0$

   (c) $y^4 + y^3 + 2y^2 + y + 1 = 0$

ANSWER TO IN-TEXT QUESTION

1. (a) $x = -1$

   (b) $x = -6$ and $x = 1$

   (c) Neither 1 nor –1 satisfies the equation, thus there are no integer roots.

5.6 The Remainder Theorem

The remainder theorem is associated with the division of polynomials. According to this theorem, if there is a polynomial P(x) which is divided by a factor (x–a), then we are left with some smaller polynomial Q(x) [degree of Q(x) is less than P(x)] and a remainder r(x). It can be expressed as:

$$P(x) = Q(x)\,(x-a) + r(x) \qquad (5.11)$$

In other way,

$$\boxed{\frac{P(x)}{(x-a)} = Q(x) + \frac{r(x)}{(x-a)}}$$

If r(x) = 0, then P(x) is completely divisible by (x – a) and hence (x – a) is a factor of P(x). When, $r(x) \neq 0$, there exist a remainder. Suppose x = a, then

$$P(a) = Q(a)\,(a-a) + r$$

$$\Rightarrow \quad P(a) = Q(a)\,(O) + r \quad \Rightarrow P(a) = r$$

Thus, we get to the conclusion that (x – a) is the factor of P(x) if and only if P(a) = 0, i.e. there is no remainder.

Example: Prove that polynomial $P(x) = 2x^3 - x^2 - 7x + 2$ has a zero at x = 2. Also factorize the polynomial.

Solution: Put x = 2 in P(x) we get

$$P(2) = 2(2)^3 - (2)^2 - 7(2) + 2 \Rightarrow 16 - 4 - 14 + 2 = 0$$

Therefore, x=2 is a root of P(x).

Also,

$$P(x) = 2x^3 - x^2 - 7x + 2 = (x - 2)(2x^2 + 3x - 1)$$

Put differently, $P(x) = 2x^3 - x^2 - 7x + 2 = 2(x - 2)(x^2 + ax + b)$.

Expanding this expression: $P(x) = 2x^3 + (2a - 4)x^2 + (2b - 4a)x - 4b$. If this equals to $2x^3 - x^2 - 7x + 2$, then $(2a - 4) = -1$, $(2b - 4a) = -7$ and $-4b = 2$. Solving this we get values of $a = 3/2$ & $b = -1/2$.

Thus, $P(x) = (x - 2)(2x^2 + 3x - 1)$

Let us understand the remainder theorem in polynomials with the following example:

Let us divide $(x^2 - x - 20)$ by $(x - 5)$, then

$$\begin{array}{rll} & x+4 & \\ x-5 \,\big) & x^2 - x - 20 & \\ (-) & x^2 - 5x & \leftarrow x(x-5) \\ \hline & 4x - 20 & \\ (-) & 4x - 20 & \leftarrow 4(x-5) \\ \hline & 0 & \text{remainder} \end{array} \qquad \because \frac{x^2}{x} = x$$

Thus, we conclude that $(x^2 - x - 20)$ divided by $(x - 5)$ equals $x + 4$.

Sometimes, instead of getting zero as remainder, we are left with a remainder. Consider the example below:

Divide $x^3 - x - 1$ by $x - 1$, then

$$\begin{array}{rll} & x^2 + x & \\ x-1 \,\big) & x^3 - x - 1 & \\ (-) & x^3 - x^2 & \leftarrow x^2(x-1) \\ \hline & x^2 - x^1 - 1 & \\ (-) & x^2 - x & \leftarrow x(x-1) \\ \hline & -1 & \text{remainder} \end{array} \qquad \because \frac{x^3}{x} = x^2$$

Thus, we conclude that, $x^3 - x - 1 = (x^2 + x)(x - 1) + (-1)$ or alternatively, $\frac{x^3 - x - 1}{x - 1} = (x^2 + x) - \frac{1}{(x - 1)}$

5.7 POWER FUNCTIONS

A power function generally takes the form f(x) = ax^k where a and k are constants, and all x > 0. Here 'k' refers to exponent of the function. The power function appears quite similar to the exponential functions, but unlike them, here in a power function the base is variable while the exponent component remains constant.

5.7.1 Rules for power functions

While using the power functions, certain rules must be taken care of :

1. $x^{p/q} = (x^{1/q})^p = (\sqrt[q]{x})^p$ where p is an integer and q is a natural number
2. $(abcd)^p = (ab)^p\,(cd)^p = a^p b^p c^p d^p$
3. $(a+b)^2 \neq a^2 + b^2$ where a$\neq$ 0,1
4. $(a-b-c)^{\frac{1}{k}} \neq a^{\frac{1}{k}} - b^{\frac{1}{k}} - c^{\frac{1}{k}}$ where 1/k ≠ 0,1
5. $(a+b)^0 = 1$
6. $x^{-a} = \frac{1}{x^a}$ and $\frac{1}{x^{-a}} = x^a$

Example: Solve the following equation:

$$y = \frac{(x^8 \times x^{-9})}{x^{-4}}$$

Solution:

$\Rightarrow$ we know that $x^8 \times x^{-9} = x^{8-9} = x^{-1}$

$$\Rightarrow y = \frac{x^{-1}}{x^{-4}} \Rightarrow x^3$$

Example: Suppose $y^2z^5 = 32$, where y and z are positive numbers. Express z in terms of y.

Solution:

$y^2z^5 = 32$

$\Rightarrow z^5 = 32\, y^{-2}$

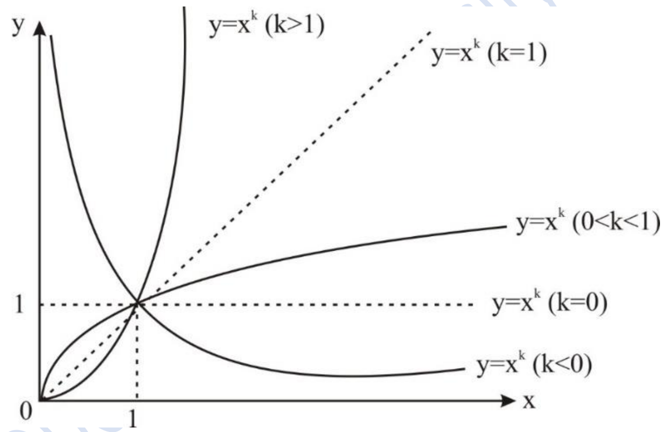
$\Rightarrow z = 32^{1/5}\, y^{-2/5} \Rightarrow z = (2)^{5/5}\, y^{-2/5}$

$\Rightarrow z = 2y^{-2/5}$

5.7.2 Graphs of power functions

Let us take the power function of the form $y = x^k$ which is defined for all rational number k and $x > 0$. Now, we will draw this graph under the following cases:

1. If $k = 0$, then $y = x^0$ and $y = 1$
2. If $k = 1$, then we have straight line $y = x$
3. If $k > 1$, then we get an upward sloping graph as the value of x increases. If x is small, the graph will be close to x-axis while the graph is vertical for large values of x.
4. For $k < 0$, then $y = x^{-k}$ or $y = \left(\frac{1}{x}\right)^k$ and we obtain downward sloping graph as the value of x increases. If x is small, the value of y increases and the graph is closer to y-axis, for large values of x, it becomes close to x-axis.
5. If $0 < k < 1$, let $k = 1/2$ then in this case, as value of x increases, the graph becomes flatter. Fig – depicts all cases.

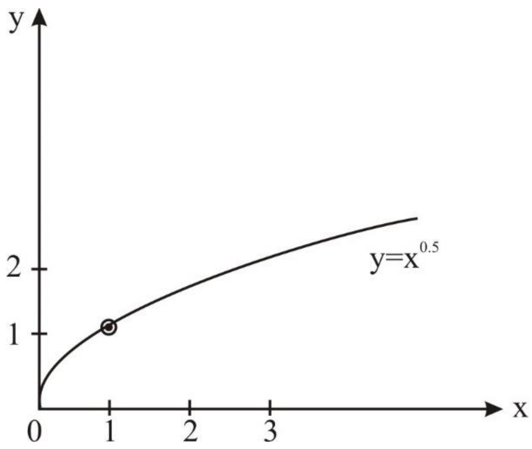


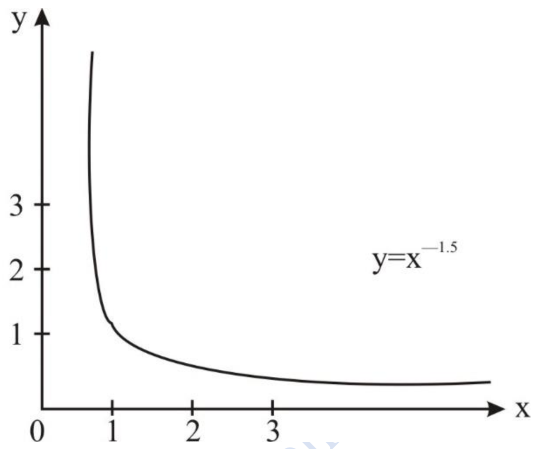
Example:

Sketch the graph of $y = x^{0.5}$ and $y = x^{-1.5}$

Solution: Using the calculator, we can show that

x	0	1/2	1	2	3
$y=x^{0.5}$	0	0.707	1	1.41	1.73
$y=x^{-1.5}$	not def.	2.82	1	0.353	0.192





IN-TEXT QUESTIONS

1. Perform the following division

   (a) $(y^2 - y - 20) \div (y - 5)$

   (b) $(3y^8 + y^2 + 1) \div (y^3 - 2y + 1)$

   (c) $(y^5 - 3y^4 + 1) \div (y^2 + y + 1)$

2. Compute the following

   (a) $32^{-3/5}$

   (b) $100^{1/5}$

   (c) $(48)^{-3/16}$

3. Simplify the expression

$$\frac{ax^{1/2} - (x - b)a^{1/2}x^{-1/2}}{(x^{1/2})^2} \qquad x > 0$$

ANSWER TO IN-TEXT QUESTION

1. (a) y + 4

   (b) $3y^5 + 6y^3 - 3y^2 + 12y - 12 + \dfrac{(28y^2 - 36y + 13)}{y^3 - 2y + 1}$

(c) $y^3 - 4y^2 + 3y + 1 - \frac{4y}{(y^2 + y + 1)}$

2. (a) 1/8

   (b) 5.511

   (c) 1/8

3. $\frac{(b+x)}{2ax^{3/2}}$

5.8 TERMINAL QUESTIONS

1. Solve the following equations by converting them into quadratic form:

   a) $\frac{y}{(y-2)(5y-4)} = -1$

   b) $\frac{1}{y-1} - \frac{1}{y+1} = 1$

2. For a given function G (z) = 72-(4+z)^2- (4-pz)^2, with p is constant. Determine the value of z for which G (z) attains largest value.

3. A piece of rope which is 40 cm long is molded into rectangle. Determine the maximum area that can be enclosed.

4. Simplify the expression: $(m^{1/3} - n^{\frac{1}{3}})\ (m^{\frac{2}{3}} + m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}})$

Answers

1. (a) 5y^2-13y+8=0, 8/5, 1

   (b) y^2-3=0

2. z=4 (p-1)/(p^2+1)

3. Let one side of rectangle be x and other side be y. Then 2x+2y=40 and y=20-x. Area (x) = x.(20-x) with maximum area is 100 cm^2.

4. m-n

5.9 SUMMARY

In this unit, we introduced different types of functions applied in economics and mathematics. We discussed the non-linear functions which are quite different from the linear functions. We studied polynomials that were raised to the power of two (quadratic functions), cubic functions that were raised to the power three and general polynomial functions raised with higher powers.

In the following section, the unit discusses the applications of quadratic functions in economics such as minimization of cost and maximization of profits by the firm and how to solve inverse demand and supply equations. In the last section of the unit, we discussed the remainder theorem wherein we looked at how division of polynomials can be done. Finally, the power functions were introduced at the end, where the base is variable number while the exponent is constant.

5.10 REFERENCES

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LESSON 6

FUNCTIONS OF ONE REAL VARIABLE-II: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

STRUCTURE

6.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to:

1. Differentiate between exponential and logarithmic functions.
2. Calculate compound interest rates

3. Understand the relationship between elasticity and logarithmic functions.

6.2 INTRODUCTION

In this chapter, we will introduce the exponential and the logarithmic functions. These functions are widely used in economic analysis such as growth of income or wealth and compound interest. We begin this chapter with exponential functions.

6.3 EXPONENTIAL FUNCTIONS

Let us define a function '*y*' such that it has a constant base 'α' and is raised to a variable component '*x*' (also known as exponent). It can be expressed as:

$$y = \alpha^x, \alpha > 0 \text{ and } \alpha \neq 1, \; x \in R \qquad (6.1)$$

This is known as the exponential function. These functions are widely used in the determination of population growth, compounding of interest rates, rates of decay and depreciation.

The exponential functions have the following general properties with $\alpha > 0$:

(1) If $x = 0$, then $y = 1$ irrespective of the base

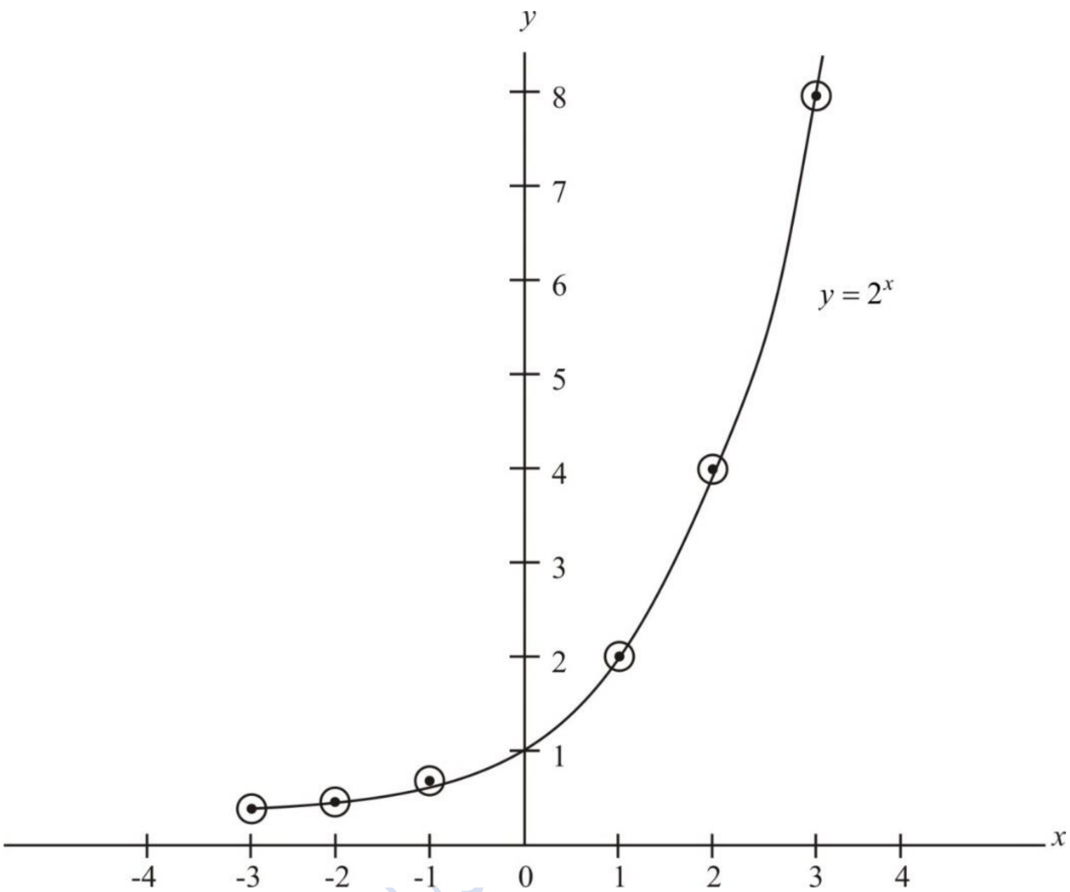
(2) Let $\alpha > 1$, the function '*y*' will be increasing and for values $0 < \alpha < 1$ the function will be decreasing.

(3) The exponent 'x' belongs to a set of real numbers, the range of 'y' is set of all positive real numbers even if $x < 0$.

Example: Let $y = 2^x$. Plot the graph of y.

Solution: To graph this function, we can simply take some value of *x* such that

x	1	2	3	0	–1	–2	–3
y	2	4	8	1	1/2	1/4	1/8



Example: (Doubling Time) In the case of population growth rate, we use a characteristic known as 'doubling time' which refers to the time required for population to double given a constant growth rate. In exponential terms, doubling time can be represented as :

$$F(t) = Kk^t \qquad \text{for } k > 1$$

where K is the size of population at time t = 0. In order to determine the doubling time t^*, let us choose an arbitrary time period t_o with population size equals K. If at time period t_1, the size of the population doubles, then it should be double the size of population in the previous period, such that: -

$$F(t_2) = Kk^{t_1} = 2K .$$

Then, doubling time $T_{double} = t_1 - t_0$ and is independent if the year chosen as base

Now,

$$Kk^{t_1} = 2K$$

Dividing both sides by k, we are left with

$$k^{t_1} = 2 \qquad \text{or} \quad k^{t^*} = 2 \qquad (*)$$

Here, the doubling time is the power to which 'k' must be raised to get value equals to 2. If the population of Uganda is growing at the rate of 4.2% then using (*), the doubling time will be

$$(b042)^t = 2$$

$$t \approx 16.85 \text{ or 17 years.}$$

Thus, it takes 17 years for population to double in Uganda.

Example (Compound Interest) if a person has a savings account of ₹A with rate of interest i% each year, then after '*t*' years it will increase to

$$A\left(1+\frac{i}{100}\right)^t$$

It i = 12% and A = ₹ ₹100, then after t years,

$$1000\left(1+\frac{12}{100}\right)^t = 100(1.12)^t$$

and if

t	1	5	10	20	30
$100(1.12)^t$	112	176.23	310.58	964.63	2995.99

IN-TEXT QUESTIONS

1. Sketch the graph of the following functions

(a) $y = 3^x$ (b) $y = 3^{-x}$ for

x	−3	−2	−1	0	1	2	3

2. How long does it take for the population to double if it is growing at the rate of 3.5%?

3. In, India, the government targets to double its per capita income over the next 10 years. What is the average annual growth rate of per capita income required to achieve this?

ANSWER TO IN-TEXT QUESTION

1.

x	**–3**	**–2**	**–1**	**0**	**1**	**2**	**3**
$y = 3^x$	1/27	1/9	1/3	1	3	9	27
$y = 3^{-x}$	27	9	3	1	1/3	1/9	1/27

2. t = 2.36

3. Per capita income is 7.18%.

6.4 NATURAL EXPONENTIAL FUNCTION

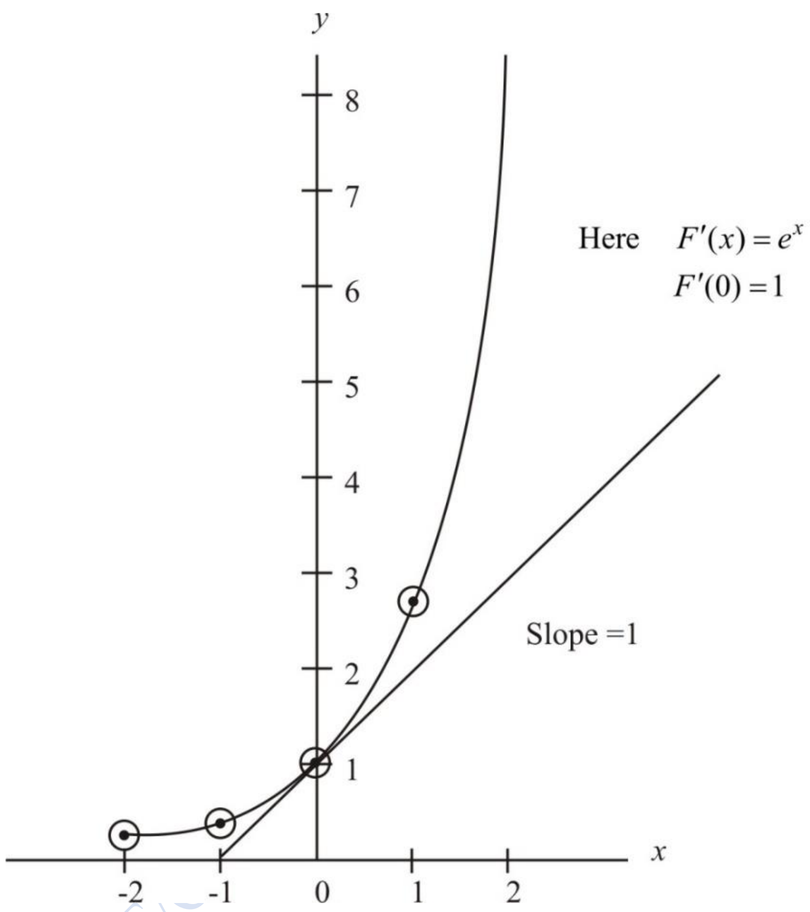
By definition, the exponential function is of the form $y = b^x$ where b is the positive number does not equal to 1. Within these exponential functions, it we take the base as an irrational number e= 2.718 then the function becomes

$$f(x) = e^x \text{ where e=2.718 is a constant}$$

Thus, an exponential function to the base 'e' is known as the natural exponential function. These functions have a property that its derivative always equals to the function itself, that is $f'(x) = e^x$.

However, computing powers with base e is difficult and cannot be done by hand. A scientific calculator is used for this purpose as it has a function key with the option e^x, that does calculation easily. Graphically, y=e^x can be shown as a strictly increasing function for all x as $e^x > 0$.

x	2	–1	0	1	2
y	0.135	0.367	1	2.71	7.38



In this case $F'(0)$ can be interpreted as slope of the lien that is tangent to the graph $y = e^x$. Thus, the slopes equal 1. It supposes the function is of the form $y = e^{h(x)}$ and we need to find y', then

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx} \quad v = h(x),\ y = e^v$$

$$\Rightarrow \qquad e^v \cdot v' \Rightarrow e^{h(x)} h'(x) \qquad (6.2)$$

Example: Find the derivative of the following:

(a) $y = e^{x^2}$ (b) $y = 3e^{7-2v}$

Solution: (a) Here $h(x) = x^2$ and $h'(x) = 2x$, then

$$y' = e^{x^2} \cdot 2x = 2xe^{x^2}$$

(b) here $h(x) = 7 - 2v$ $\qquad h'(v) = -2$

then $y' = 3e^{7-2v} \cdot (-2) = -6e^{7-2v}$

6.5 PROPERTIES OF EXPONENTIAL FUNCTION WITH BASE e

1. The natural exponent function $f(x) = e^x$ is strictly increasing and differentiable for all real numbers x, such that $f'(x) = e^x$.
2. Let *a* and *b* be two exponents, then –
   (i) $e^a e^b = e^{a+b}$
   (ii) $\dfrac{e^a}{e^b} = e^{a-b}$
   (iii) $(e^a)^b = e^{ab}$

Example: Compute the following:

(a) $e^{5v} \cdot e^{2x}$ $\qquad$ (b) $\dfrac{e^{4y}}{e^{7y}}$

Solution: (a) $e^{5y} \cdot e^{2x} = 3^{5y+2x}$

(b) $\dfrac{e^{4y}}{e^{7y}} = e^{4y-7y}$

$$= e^{-3x} \Rightarrow \frac{1}{e^{3x}}$$

6.6 LOGARITHMIC FUNCTIONS

In the previous section we discussed the function '*f*' of the form $y = \alpha^x$. If we interchange the variables of this function, then we get a new function as 'g' as $x = \alpha^y$. Here if $f(1) = 2$ then $g(2) = 1$. This inverse function '*g*' of the exponential function '*f*' is known as logarithmic function with base α. It can be expressed as:

$$\text{Y=log}\alpha^x \qquad \alpha > 0, \alpha \neq 1 \qquad (6.3)$$

These functions are strictly increasing and monotonic and also concave everywhere. In general terms, a logarithmic function defined as y= logα^x it is read as '*y* is the base *a* logarithm of *x*'.

Logarithmic transformation of the models' variables is frequently used in the economic models. A logarithmic transformation is the conversion of a variable into its logarithm which can take on various real positive values. In our case, any positive number except $a = 1$ can be the base for a logarithm. Most often, we come across $\log x$, which is read as exponent to which 10 must be raised to get 'x'. Like exponential functions, logarithmic functions have certain properties.

(1) The domain of the function is the set of all positive real numbers. The range of the function is a set of real numbers.
(2) Given the base 'a', if $a > 1$ then the logarithmic function is increasing and for $0 < a < 1$, it is decreasing.
(3) Also, $y = 0$ if $x = 1$, independent of the choice of base.

When using the base for a logarithmic function, an irrational number 'e' is often used, that is

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718$$

For example, $e^x = 14$

$$2e^{-2x} = 8$$

Here, exponential functions written as $b = e^x$ can be expressed as $x = \ln b$ which we termed as the natural logarithmic functions. Here, $\ln b$ is the power of e need to get b. However, when using these functions, certain points have to be kept in mind:

(i) If 'e' is raised to any variable $x\ (x > 0)$ or a constant 'b' (b > 0), then function of this variable must equal to variable or constant

such as $e^{\ln b} = b$

$e^{\ln x} = x$

(ii) The natural log of 'e' raised to the power of a variable or constant, must equal that variable or constant, such as:

$$\ln e^b = b \quad and\ \ln e^x = x$$

6.6.1 Properties of Logarithmic Functions

Let y and z be positive real numbers, and b ≠ 1

(1) $\log_b yz = \log_b y + \log_b z$

(2) $\log_b \frac{y}{z} = \log_b y - \log_b z$

(3) $\log_b y^k = k \log_b y$

(4) $\log_b \sqrt[k]{y} = \frac{1}{k}\log_b y$

(5) $\ln 1 = 0$ (6) $\ln e = 1$

(7) $\ln(1/e) = \ln e^{-1} = -1$

(8) $\log_b a = \frac{\log_x a}{\log_x b}$, $x \in R$

Example: Solve the equation for *x*:

(a) $4e^{x+2} = 120$

(b) $1/2e^{2} = 144$

(c) $e^{x} + e^{-x} = 2$

Solution:

(a) $4e^{x+2} = 120$

$\Rightarrow e^{x+2} = 30$

Taking natural log on both sides

$\ln e^{x+2} = \ln 30$

$\Rightarrow (x+2) = \ln 30$ (From properties)

$\Rightarrow x = \ln 30 - 2$

(b) $1/2e^{x^2} = 144$

$\Rightarrow e^{x^2} = 288$

Taking log on both sides

$\ln e^{x^2} = \ln 288$

$\Rightarrow x^2 = \ln 288$

(c) $e^{x} + e^{-x} = 2$

If we take in both sides, then

$\ln(e^{x} + e^{-x}) = \ln 2$

But this cannot be evaluated. Suppose $p = e^{x}$ and $\frac{1}{p} = e^{-x}$, then

$\Rightarrow p + \frac{1}{p} = 2 \Rightarrow p^2 + 1 = 2$

$\Rightarrow p^2 - 2p + 1 = 0$

On solving this, we get p= 1 and hence $e^{x} = 1$ and thus x = 0.

6.6.2 Inverse Function g(x)

If suppose x belongs to positive real number and we defined, $e^{\ln x} = x,$ then the function $g(x)$ expressed as

$$g(x) = \ln x \qquad (x > 0) \qquad (6.4)$$

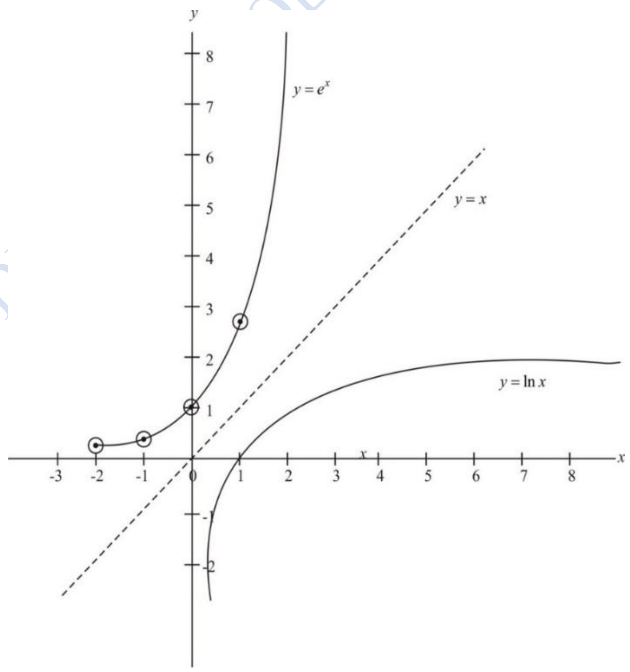
in such case one function will be the mirror image of another function. Let us take two functions. $f(x) = e^x$ and $g(x) = \ln x$. If we graph these two functions, then

x	-2	-1	0	1	2
$f(x) = e^x$	0.135	0.367	1	2.71	7.38

and let $y = g(x) = \ln x$, then

y	-2	-1	0	1	2
$g(x)$	0.135	0.367	1	2.71	7.38

As, observe one function is the inverse of the other function



From this graph, three conclusions can be drawn:

(i) The domain of $f(x)$ is range of $g(x)$, while the range of $f(x)$ is the domain of $g(x)$
(ii) $g(x)$ is increasing from positive to negative value and is strictly increasing with domain (0, ∞).
(iii) $\ln x$ is positive for x>1 and negative for 0<x<1.

6.6.3 Differentiation and Logarithmic Function

Let us define a function $f(x) = e^{p(x)}$, where $p(x)$ has a derivative for all $x > 0$, then

$$f'(x) = e^{p(x)} \cdot p'(x) \qquad (6.5)$$

Hence, the derivative of an exponential function will equal to the initial exponential function times the derivative of that exponent.

For example: $f(x) = e^{x^2}$

Here $p(x) = x^2$ and $p'(x) = 2x$, then

$$f'(x) = e^{x^2} \cdot 2x \Rightarrow 2xe^{x^2}$$

Alternatively, if $f(x) = \ln p(x)$, where $p(x)$ is differentiable and positive, then using the chain rule,

$$f(x) = \frac{1}{p(x)} \cdot p'(x) = \frac{p'(x)}{p(x)}$$

For example: $f(x) = \ln 3x^2$, here $p(x) = 3x^2$ and $p'(x) = 6x$.

$$\Rightarrow f'(x) = \frac{1}{p(x)} \times p'(x) = \frac{1}{3x^2} \times 6x = \frac{2}{x}$$

Example: Find the derivative of the function

$$h(x) = \frac{(4x^3 - y)(3x^4 + 7)}{(9x^5 - 2)}$$

Solution: Taking natural log on both sides, we obtain

$$\ln h(x) = \frac{(4x^3 - y)(3x^4 + 7)}{(9x^5 - 2)}$$

$$= \ln(4x^3 - y) + \ln(3x^4 + 7) - \ln(9x^5 - 2)$$

We know that $\ln h(x)$ derivative is $\frac{h'(x)}{h(x)}$, then derivative of

$$\ln(4x^3 - 7) = \frac{1}{(4x^3 - 7)} \times 12x^2$$

We will obtain derivatives for other terms similarly, thus

$$h'(x) = \frac{12x^2}{(4x^3 - 7)} + \frac{12x^3}{3x^4 + 7} - \frac{45x^4}{9x^5 - 2}$$

So, finally $\frac{h'(x)}{h(x)}$ becomes

$$h'(x) = \left[\frac{12x^2}{(4x^3 - 7)} + \frac{12x^3}{3x^4 + 7} - \frac{45x^4}{9x^5 - 2}\right] \times \frac{(4x^3 - 7)(3x^4 + 7)}{(9x^5 - 2)}$$

Example: Compute the derivative and find the domain

$$y = \ln(1 - 2x)$$

Solution: In this case $\ln(1 - 2x)$ will be defined only if $y = 1 - 2x > 0 \Rightarrow x < 1/2$. Using the above case, let

$g(x) = 1 - 2x$, then $g'(x) = -2$. Thus,

$$y' = \frac{1}{(4 - 2x)} \times (-2) = \frac{-2}{(1 - 2x)}$$

Example: Simply the expression

(a) $\exp[\ln(y)] - \ln[\exp(y)]$

(b) $\ln[y^4 \exp(-y)]$

Solution: (a) It can be written as $e^{\ln y} - \ln e^y$. From the properties discussed in previous section:

$$e^{\ln y} = y \quad \text{and} \quad \ln e^{y} = y$$

Thus $\quad y - y = 0$

(b) $\ln[y^4 e^{-y}]$. We knew that $\ln xy = \ln x + \ln y$. Thus,

$\Rightarrow \quad \ln y^4 + \ln e^{-y}$

$\Rightarrow \quad 4\ln y - y$

IN-TEXT QUESTION

1. Express the following in terms of ln5

   (a) ln25 (b) $\ln\sqrt{5}$ (c) $\ln\sqrt[5]{5^2}$

2. Solve the following equations for y.

   (a) $\log_4 y = 3$ (b) $\frac{y\ln(y+3)}{y^2+1} = 0$

   (c) $\ln(\sqrt{y} - 5) = 0$ (d) $\ln(y^2 - 4y + 5) = 0$

3. Solve the following equations for 'k'

   (a) $4e^{3k-1.5} = 360$ (b) $\frac{1}{2}e^{k^2} = 259$ (use calculator)

4. Find the domain of the following function

   $f(x) = \ln(x+1)$ (b) $f(y) = \ln\left(\frac{3y-1}{1-y}\right)$

   (c) $f(y) = \ln(\ln y)$

5. Differentiate each of the following logarithmic functions

   (a) $\ln(1+y)$ (b) $\ln(e^x + 1)$ (c) $\ln(x+4)^2$

6. Let the production function is defined as follows:

$$H(\beta) = 0\left(\frac{L^{\beta}K^{\beta}}{L^{\beta} + bK^{\beta}}\right)^{c/\beta}$$

(Where *a*, *b*, *c*, *L* and *K* are positive). Find $H'(\beta)$

ANSWER TO IN-TEXT QUESTION

1. (a) $\ln 25 = \ln 5^2 = 2\ln 5$ (b) $\frac{1}{2}\ln 5$ (c) $\ln 5^{2/5}$

2. (a) $y = 64$ (b) $y = 0$ or $\ln(y+3) = 0$ so $y = 0$ or -2

   (c) $\sqrt{y} - 5 = 1 \Rightarrow y = 36$ (d) y =2

3. (a) $k = \frac{\ln 90 + 1.5}{3}$ (b) $K^2 = 6.25 \quad K = \pm 2.5$

4. (a) $x > -1$ (b) $1/3 < y < 1$ (c) $y > 1$

5. (a) $\frac{1}{1+y}$ (b) $\frac{e^x}{e^x + 1}$ (c) $\frac{2}{(x+4)}$

6. Take the logarithm of whose expression

$$\ln a + \left(\frac{c}{\beta}\right)[\beta \ln L + \beta \ln K - \ln(L^{\beta} + bK^{\beta})$$

$$F'(\beta) = \alpha\beta^{-2}F(\beta)\left(\ln(L^{\beta} + bK^{\beta}) - \beta\frac{L^{\beta}\ln L + bK^{\beta}\ln K}{L^{\beta} + bK^{\beta}}\right)$$

6.7 POWER FUNCTION AND LOGARITHMS

In Chapter 5, we have introduced power functions of the form $f(x) = x^b$ for all $b \in R$ On differentiating, we obtained $f(x) = bx^{b-1}$.

In a case, if b is an 'irrational' number then for all $x > 0$, we can define

$$x = e^{\ln x}$$

$$x^b = (e^{\ln x})^b = e^{b \ln x} \quad (6.6)$$

Differentiating (6.6) using the chain rule, we get

$$\frac{d}{dx}(x^b) = \frac{d}{dx}(e^{b\ln x}) = e^{b\ln x} \cdot \frac{b}{x} = 2^b \cdot \frac{b}{x} \Rightarrow bx^{b-1} \quad (6.7)$$

Hence, when 'b' is an irrational number, it can be differentiated.

6.8 LOGARITHMIC FUNCTION RULE FOR BASE OTHER THAN 'e'

In the previous section we defined an exponent with the base '*e*'. However, the base of an exponent can also be a fixed positive number. Let '*b*' be that number ($b > 1$), and $b^{\alpha} = x$. This implies that α is the logarithm of *x* to the base *b*. In other words,

$$\alpha = \log_b x$$

Given this, we can define $\log_b x$ for every positive number 'x' such that:

$$b^{logx} = x$$

If we take log on both sides, then

$$\log_b x \,.\log b = \log x$$

$$\Rightarrow \qquad \log_b x = \frac{1}{\log b} \times \log x \quad (6.8)$$

Here, the logarithm of x with base '*b*' is proportional to ln*x* and multiplied by the factor $1/\ln b$. This follows the same rule as in the case of natural logarithms, Hence, the rules for an exponent with a positive number '*b*' is

(1) $\log_b(xy) = \log_b x + \log_b y$

Proof: $\log_b(xy) = \frac{1}{\log b} \times \log(xy)$ (From *)

$$= \frac{1}{\log_b}[\log x + \log y] = \frac{1}{\log_b} \times \log x + \frac{1}{\log_b} \times \log y$$

$$= \log_b x + \log_b y$$

Similarly, other properties: (2) $\log_b \frac{x}{y} = \log_b x - \log_b y$

(3) $\log_b x^k = k \log_b x$

(4) $\log_b 1 = 0$ (5) $\log_b b = 1$

The function $f(x)$ defined as $f(x) = \log_b h(x)$ with b > 0 and b ≠ 1 and $h(x)$ is differentiable and positive then in that case,

$$f'(x) = \frac{1}{h(x)} \times h'(x) \cdot \log_b e$$

$$\Rightarrow \boxed{f'(x) = \frac{1}{h(x)} \times h'(x) \cdot \frac{1}{\log_b}}$$

Example: Find the derivative of $f(x) = \log_b (3x^2 + 1)$.

Solution: Here $h(x) = 3x^2 + 1$ and $h'(x) = 6x$. From the above formulae.

$$f'(x) = \frac{1}{3x^2 + 1} \times (6x) \times \frac{1}{\ln b}$$

or $$f'(x) = \frac{6x}{(3x^2 + 1)\ln b}$$

Example: Find the derivative of $y = \log_b x$

Solution: In this case $h(x) = x$ and $h'(x) = 1$. Thus

$$y' = \frac{1}{h(x)} \times h'(x) \times \ln_b e$$

$$y' = \frac{1}{x} \times (1) \times \frac{1}{\ln b} \Rightarrow \frac{1}{x \ln b}$$

IN-TEXT QUESTIONS

1. Simplify the following expression using properties of logarithmic functions

   (a) $\log_a \frac{y^7}{x^4}$ (b) $\log_a \sqrt[3]{y}$ (c) $\log_a x^5 y^{-4}$

2. Find the derivative of the following:

   (a) $F(y) = \log_b(4y^2 - 3)$ (b) $G(x) = \log_b \sqrt{x^2 - 7}$

ANSWER TO IN-TEXT QUESTION

1. (a) $7\log_a y - 4\log_a x$ (b) $1/3\log_e y$ (c) $5\log_a x - 4\log_a y$

2. (a) $\frac{8y}{(4y^2-3)\ln b}$ (b) $\frac{x}{(x^2-y)\ln b}$

6.9 APPLICATIONS OF EXPONENTIALS AND LOGARITHMS

In economics, both exponentials and logarithms are often used to estimate the growth rates, elasticity of demand and supply, simplification of non-linear functions and interest rate compounding. In this section, we will look into these applications

DETERMINATION OF GROWTH RATE

Let us define the growth function as $G = x = g(t)$ such that

$$G = \frac{g'(t)}{g(t)} = \frac{x'}{x} \qquad (6.9)$$

This function can be determined by dividing the derivative of the function with the function itself or by taking logarithm on both sides and then differentiating that function.

Example: Find the growth rate of $A = Ke^{rt}$ where K is constant.

Solution: By first method, using (6.9)

$$G = \frac{A'}{A} \Rightarrow A' = Ke^{rt}(t) = rKe^{rt}$$

Thus, $\frac{A'}{A} = \frac{rKe^{rt}}{Ke^{rt}} = r$

Also, by taking ln on both sides

$$\ln A = \ln K + \ln e^{rt}$$

$$\Rightarrow \ln A = \ln K + rt$$

Taking derivative of this function with respect to t, then

$$G = \frac{1}{A}\frac{dA}{dt} = \frac{d}{dt}(\ln A)$$

$$= \frac{d}{dt}(\ln K + rt) = 0 + r = r$$

Example: Find the growth rate of profit at t = 8 given

$$P(t) = 250000e^{1.2t^{1/3}}$$

Solution: Taking log on both sides

$$\ln P(t) = \ln 250000 + \ln e^{1.2t^{1/3}}$$

Taking derivative w.r.t. 't' on both sides

$$G = \frac{d}{dt}(\ln P(t)) = \frac{P'(t)}{P(t)}$$

$$\ln P(t) = \ln 250000 + 1.2t^{1/3}$$

$$\Rightarrow \frac{d}{dt}(\ln 250000 + 1.2et^{1/3})$$

$$\Rightarrow 0 + 1.2\left(\frac{1}{3}\right)t^{1/3-1} = \frac{0.4}{t^{2/3}}$$

with $t = 8 \Rightarrow G = \frac{0.4}{(8)^{2/3}} = \frac{0.4}{4} = 0.1$ or 10%.

LOG-LINEAR RELATIONS

In economic models, sometimes non-linear functions are converted into linear ones using the logarithmic functions. Let us suppose there are two variables *u* and *v* defined as

$v = Au^a$ (*A*, *u*, *v* is positive)

This non-linear function can be converted into linear form by taking logarithm to any base on both sides such that:

$$\log v = \log A + a \log u \qquad (6.10)$$

This transformation is known as *log-linear* relation between *v* and *u*.

Example: Take the case of Cobb-Douglas production function

$$Q = BK^{\beta} L^{\beta}$$

Can be re-written in the log-linear form as

$$\ln Q = \ln B + \ln K^{\beta} + \ln L^{\beta}$$

$$\Rightarrow \qquad \ln Q = \ln B + \alpha \ln K + \beta \ln L$$

ELASTICITIES AND LOGARITHMIC FUNCTIONS

Logarithmic functions are often used to determine elasticity. If we define the demand function as $y = D(P)$, then elasticity of this function with respect to P is $\frac{p}{D(p)} = \frac{dD(p)}{dp} = El_p D$

Suppose $D(p) = 400p^{-2}$,then using the formulae we get

$$\frac{p}{400p^{-2}} \times 400(-2)p^{-3} = -2$$

Other way, taking log on both sides, we get

$$\ln D(p) = \ln 400 - 2\ln p$$

Taking derivative w.r.t. p.

$$\frac{d \ln D(p)}{d \ln p} = -2 \qquad \text{(Slope of this log-linear function)}$$

'Thus, in general if u and v are two positive variables with v is differentiable function u, then

$$E \ln v = \frac{u}{v}\frac{dv}{du} = \frac{d \ln v}{d \ln u} = \frac{d \log_b v}{d \log_b u} \qquad (6.11)$$

where, b is any positive base.

Example: Find the elasticity of the function

$$y = e^x$$

We know that $E \ln y = \frac{x}{y} \times \frac{dy}{dx}$

or taking logs on both sides

$$\ln y = \ln e^x$$

$$\rightarrow \ln y = x$$

Using chain rules and differentiation,

$$\frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dv}{dx} = y\,]$$

Now, $Elxy = \frac{x}{y} \times y = x$

hence, the elasticity equals x.

IN-TEXT QUESTIONS

1. For country X, the national income (Y) is increasing at the rate of 1.5% per year while the population (P) is growing at the rate of 2.5%. Determine the per-capital rate of growth of income [Hint: Find 1/*p*]

2. Estimate the growth rate of sales (S) if

   $S(t) = 1,00,000e^{0.5\sqrt{t}}$ with t = 4.

3. The number of people (P) who develop corona 't' days after a group of 1000 people has been in contact with corona infection is given by:

$$P(t) = \frac{1000}{1 + 999e^{0.39t}}$$

(a) How many people develop corona after 20 days?

(b) Estimate the number of days when 800 people are side?

4. Find the elasticity of $y = f(x)$ with respect to x:

(a) $y = e^3 e^{2x}$ (b) $y = x\ln(x+1)$

5. Given the log-linear relationship. $y = 594{,}500u^{-0.3}$, or press u terms of y.

ANSWER TO IN-TEXT QUESTION

1. There will be a fall in per-capita income by 1%

2. 12.5%

3. (a) 710 (b) About 21 days

4. (a) 3 + 2 x (b) $1+\frac{x}{(x+1)\ln(x+1)}$

5. $\ln y = \ln 694500 - 0.3\ln u \Rightarrow \mu = \left(\frac{644500}{2}\right)^{10/3}$

6.10 COMPOUND INTEREST AND PRESENT DISCOUNTED VALUES

Exponential functions can be used in the applications of interest rate compounding. Let us suppose a principal amount ₹ A is compounded annually with interest rate i% for a time period 't' years. Then at the end of this time period, we will receive an amount ₹P given by the exponential function as

$$P = A(1+r)^t \quad \text{where } r = i/100$$

If it is compounded in times a year for 't' years, then

$$= A\left(1+\frac{r}{m}\right)^{mt}$$

here the principal is multiplied by a factor $(1 + r/m)^m$ each year. If it is compounded at 100% interest rate for one year, then:

$$P = A\left(1+\frac{1}{m}\right)^m \qquad (6.12)$$

If m $\rightarrow \infty$ then, $A\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = A(2.718) = Ae$

Thus, for any interest rate i% and time period t (t $\neq$ 1)

$$P(t) = Ae^{rt} \qquad (6.13)$$

This is known as continuous compounding with 'r' as the rate of interest. If we differentiate (6.13), then we obtain

$$\frac{P'(t)}{P(t)} = r$$

the principal increases at the constant rate 'r' with the continuous compounding of the interest rate.

Example: Find value of ₹100b compounded continuously at an interest rate of 10% for two years.

Solution: For continuous compounding, $P = Ae^{rt}$

$\Rightarrow \quad P = 100e^{0.1(2)},\ r = 0.1, t = 2$

$\Rightarrow \quad P = 100e^{0.2}$

$\Rightarrow \quad P = 100(1.221) = 122.1$ (Using calculation)

6.10.1 Effective and Nominal Interest Rates

In the previous example when interest rate was compounded continuously, we earned ₹122.1, if we did it annually, we must have got ₹121 and semi-annually, that is $100\left(1+\frac{0.1}{2}\right)^{2(2)} = 121.55$. When an individual borrows money from any financial institution, he must compare the various options. In this regard an 'effective rate of interest' helps an individual in making such comparisons. Let us define the effective rate of interest as 'r_e' which compounded continuously gives the same total interest rate over the year such that

$$A(1+r_e)^t = A\left(1+\frac{r}{m}\right)^{mt} \qquad r = i/100$$

Dividing both sides by A

$$(1+r_e)^t = \left(1+\frac{r}{m}\right)^{mt}$$

Taking the 't' root on both sides, we get

$$(1+r_e) = \left(1+\frac{r}{m}\right)^{m} \Rightarrow \boxed{r_e = \left(1+\frac{r}{m}\right)^{m} - 1}$$

We know that $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = e$

$$\Rightarrow \qquad (1+r_e) = e^r \qquad \text{(From 6.13)}$$

$$\Rightarrow \qquad \boxed{r_e = e^r - 1} \qquad (6.14)$$

This is the case when compounded continuously.

Example: Find the effective annual interest rate on ₹100 with 12% interest rate when it is compounded quarterly and continuously.

Solution: Here, A = ₹1000, r = 0.12, m = 4

$$r_e = \left(1+\frac{r}{m}\right)^{m} - 1$$

$$r_e = (1+0.12)^4 - 1 \Rightarrow (0.13)^4 - 1$$

$$r_e = 12.55\%$$

Thus, the yearly interest rate of 12% corresponds to an effective annual interest rate of 12.55%.

Similarly, when compounded continuously

$$r_e = e^r - 1 \Rightarrow e^{0.12} - 1$$

$$= 1.127 - 1 = 0.127 \text{ or } 12.7\%.$$

6.10.2 Present Value / Discounting

When a sum of money is deposited in a bank, then in future one receives an amount not equivalent to the money in the present, but even a large amount at the end of the year. It at present, a person deposits a sum of ₹100 at 12% interest compounded annually, then he must receive a sum of ₹120 one year from new, and thus the 'present value' of this amount is ₹100 today. In general terms, if P is the amount deposited today with interest rate of r% per year for t years, and we get an amount A after t years then

$$P\left(1 + \frac{r}{100}\right)^t = A$$

$\Rightarrow$ $\quad P(1+i)^t = A$ $\quad$ where i = r/100

Then $\quad P = A(1+i)^{-t}$ with yearly interest rates

and under continuous compounding present value becomes

$$P = Ae^{-rt} \qquad (6.15)$$

This process of finding the present value P of the future payment is known as discounting.

Example: Find the present value of ₹500 to be paid in 3 years at 8% interest rate compounded continuously.

Solution: For continuous compounding of interest rate:

$$P = 500e^{-0.08(3)}$$

$$\Rightarrow \quad P = 500e^{-0.24}$$

Using calculator, $e^{-0.24} = 0.786$

$$\Rightarrow \quad P = 500(0.786) = 393.32$$

IN-TEXT PROBLEMS

1. Compute the future value ₹100 for 6 years at 5 percent interest rate when compounded (a) continuously (b) annually.
2. Find the present value of ₹120 to be paid in 5 years at 9 percent interest rate compounded (a) annually (b) continuously.

ANSWER TO IN-TEXT QUESTION

1. (a) $100e^{0.3}$ or 134.99 (b) 134.01

2 (a) 77.99 (b) $120e^{-0.45}$ or 76.52

6.11 TERMINAL QUESTIONS

1. Determine the elasticity of supply for a competitive firm with supply function given as:

$$Q = Mp^m + Np^n$$

Where M, N,m,n are positive and m>n.

2. Suppose population size P and aggregate wealth W is expressed as:

$W = c+at$, $P = Le^{mt}$

Where c,a, L and m are constants. Determine continuous growth rate of population, wealth and wealth per head.

3. Differentiate the following:

a) $\ln(x^4+1)$

b) $x/(1+e^x)$

4. find the inverse g(y) of the following function:

$$f(y)=\begin{cases} 2y, & y \le 0 \\ y^2 & y > 0 \end{cases}$$

Also determine domain of the inverse function

Answers:

1. $Mmp^m + Nnp^n / Mp^m + Np^n$

2. Population size= m, wealth= $\frac{a}{c+at}$ and wealth per head= $\frac{a}{c+at} - m$

3. (a) $4x^3/x^4+1$ (b) $1+(1-x)e^x/(1+e^x)^2$

4. The inverse function

$$G(y) = \begin{cases} y/2, & y \leq 0 \\ \sqrt{y}, & y > 0 \end{cases}$$

6.12 SUMMARY

In this unit, we discussed the exponential functions in which the independent variable takes the form of an exponent. A fixed base is raised to a variable exponent under the exponential functions. Additionally, the unit also covered the logarithmic functions. They are the inverse of exponential functions.

The second part of the unit discusses the economic applications of exponential and logarithmic functions. We have discussed how these functions can be applied to estimate elasticity and growth rate of a variable over time. Lastly, we conclude this unit with the applications of exponential function in interest rate compounding.

6.13 REFERENCES

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LESSON 7

SEQUENCES, SERIES AND LIMITS

Structure

7.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to:

1. Define the limit of a function
2. Find the limits of sequences and series

3. Determine the present and compound values of income streams and
4. Understand the decision on how to make investments.

7.2 INTRODUCTION

In this unit, we will introduce the theoretical concept of limits and continuity. We will also discuss how economics can be used financially to ascertain the payment of interest, present discounted value and compounding of interest.

7.3 LIMITS

We introduced the concept of limit in previous unit. In this section, we will further look at the concept of limits by understanding the case of one-sided limits and limits at infinity.

One-Sided Limits

Let us consider a function $f(h)$. If h tends to 'a', then $f(h)$ might tend to value 'A' for all h sufficiently close to 'a'.

Mathematically,

$$\lim_{h \to a} f(h) = A \quad \text{or} \quad f(h) \to A \quad \text{as} \quad h \to a \qquad (7.1)$$

However, there exist ways in which h can tend to value 'a'. In one way h can tend 'a' from values smaller than 'a' known as ***left hand side*** or it can tend from values greater than 'a', the ***right-hand side.***

In other words, if h tends from left hand side, then $f(h)$ tends to value 'K', that is limit of $f(h)$ tends to 'a' from below is 'K' represented as

$$\lim_{h \to a^-} f(h) = K \text{ or } f(h) \to K \quad \text{as} \quad h \to a^- \qquad (7.2)$$

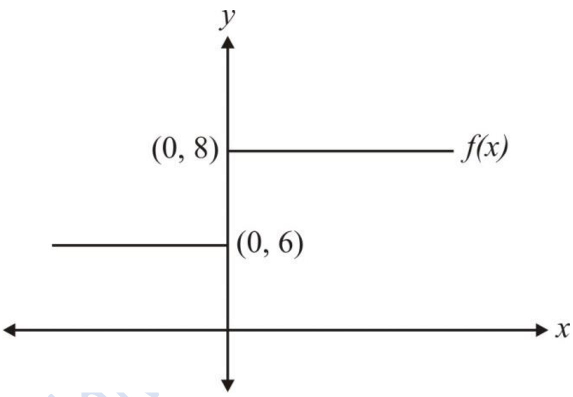
Similarly, if h tends from right hand side, then $f(h)$ tends to value A that is limit of $f(h)$ as h tends to a from above is A and expressed as :

$$\lim_{h \to a^+} f(h) = A \text{ or } f(h) \to A \text{ as } h \to a^+ \qquad (7.3)$$

The above two cases are referred to as 'one-sided' limits.

Example: Let $f(x) = \begin{cases} 6, & x \le 0 \\ 8, & x > 0 \end{cases}$

Solution:



As $x \to 0^-$, then $f(x) \to 6$

$$\lim_{x \to 0^-} f(x) = 6$$

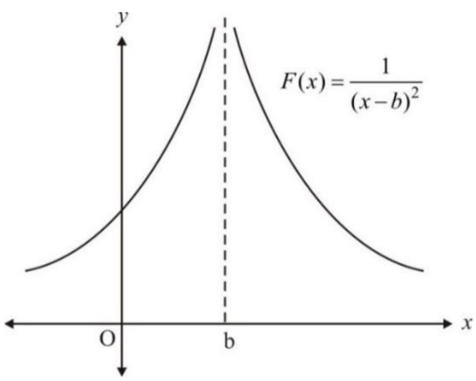
White x tends from above as $x \to 0^+$ then $f(x) \to 8$

$$\lim_{x \to 0^+} f(x) = 8$$

For a function to have a limit, it has to satisfy the necessary and sufficient condition that if left-hand side and right-hand side limits exist, then they should be equal to each other, expressed as:

$$\lim_{h \to 0^-} f(h) = A \Leftrightarrow \lim_{h \to 0^+} f(h) = A$$

In a special case, if $f(h)$ tends to $+\infty$ or $-\infty$, then



$$\lim_{h \to 0^-} f(h) = \infty(-\infty) \qquad \lim_{h \to 0^+} f(h) = \infty \text{ (or } -\infty)$$

Then, we say that limit does not exist.

Suppose $y = f(x) = \dfrac{1}{(x-b)^2}$

then, $\lim_{x \to b} f(x) = \infty$

referred to as vertical asymptote.

In this case a limit does not exist.

Graphically also, we can depict that the limit does not exist.

7.4 LIMITS AT INFINITY

Let there exist a function $Y = f(x)$ and a real number L. As 'x' becomes sufficiently large, the values of $f(x)$ becomes close to L, then we say that $f(x)$ has a limit at infinity

$$\lim_{x \to \infty} f(x) = L \text{ and } \lim_{x \to \infty^-} f(x) = L \qquad (7.4)$$

this will hold for all $x < 0$ and x being sufficiently large. It is also known as horizontal asymptote. Thus, if $f(x)$ approaches to L as $x \to \infty$ or $x \to -\infty$, then the graph of $f(x)$ approaches the line $y = L$. This horizontal line is known as *'asymptote'* for the graph $f(x)$.

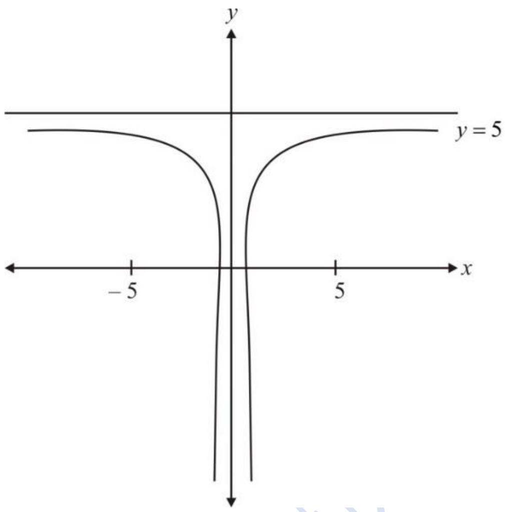
For example, $y = F(x) = 5 - \frac{2}{x^2}$ and

$x \to \infty$ as well as $x \to -\infty$

In both cases

$$\lim_{x \to \infty^+} f(x) = 5$$

$$\lim_{x \to \infty^-} f(x) = 5$$



Thus, it will have a horizontal asymptote at $y = 5$.

Example: Examine the limit if $x \to \infty$ and $x \to -\infty$

(a) $f(x) = \frac{x+1}{2x}$ (b) $f(x) = \frac{1-x^5}{x^4+x+1}$

Solution: (a) If we put $x = \infty$ then $\frac{\infty}{\infty}$ cannot be calculated. Thus, we will divide both numerator and denominator by the highest power of 'x' such that

$$f(x) = \frac{1+\frac{1}{x}}{2} \qquad \lim_{x \to \infty^+} f(x) = \frac{1+\frac{1}{\infty}}{2} = \frac{1}{2}$$

While $\lim_{x \to \infty} f(x) = \frac{1}{2}$. So, the limit exists.

(b) $f(x) = \frac{1-x^5}{x^4+x+1}$. Dividing both numerator and denominator by x^5, we get

$$F(x)=\frac{\frac{1}{x^5}-1}{\frac{1}{x}+\frac{1}{x^4}+\frac{1}{x^5}} \Rightarrow \lim_{n\to\infty} f(x)=\frac{-1}{0}=-\infty$$

$$\Rightarrow \lim_{n\to-\infty} f(x)=\frac{-1}{0}=-\infty$$

Hence, in this case limit does not exist.

RULES FOR LIMIT

From the previous chapter, we learned about the algebraic operations on limits such as addition, subtraction, but in the case, x tends to infinity: $x\to\infty$ or $x\to-\infty$, then the corresponding limit properties also change.

Let $f(x)$ and $g(x)$ be two functions with both tend to ∞ as x as x → h, where h is any real number,

$$\lim_{x\to h} f(x)=\infty \quad \text{and} \quad \lim_{x\to h} g(x)=\infty$$

then the following rules are applicable.

1. $\lim_{x\to h}[f(x)+g(x)]=\infty$

2. $\lim_{x\to h}[f(x)\cdot g(x)]=\infty$

3. $\lim_{x\to h}[f(x)-g(x)]=\infty$ = indeterminate case. No solution

4. $\lim_{x\to h}\left[\frac{f(x)}{g(x)}\right]=\frac{\infty}{\infty}$ Indeterminate case. No solution.

Similarly, if $\lim_{x\to h} f(x)=\infty$ and $\lim_{x\to h} g(x)=\infty$

5. $\lim_{x\to h}[f(x)+g(x)] = A+\infty = \infty$ Indeterminate Form

6. $\lim_{x\to h}[f(x)-g(x)] = A-\infty = -\infty$ Indeterminate form

7. $\lim_{x\to h}\left[\frac{f(x)}{g(x)}\right] = \frac{A}{\infty}$

8. $\lim_{x\to h}[f(x)\cdot g(x)] = A\times\infty = \infty$ = indeterminate case. No solution

Example: Let $f(x) = x^2 - 7x$ and $g(x) = x$ .If $x \to 0$, then determine:

(a) $f(x)/g(x)$ (b) $f(x)-g(x)$ (c) $f(x)\cdot g(x)$

Solution: (a) $\frac{f(x)}{g(x)} = \frac{x^2-7x}{x} = x-7$

$$\lim_{x\to 0} x-7 = -7$$

(b) $f(x)-g(x) = x^2-7x-x = x^2-8x$

$$\lim_{x\to 0} x^2-8x = 0$$

(c) $f(x)\cdot g(x) = (x^2-7x)x = x^3-7x^2$

$$\lim_{x\to 0}(x^3-7x^2) = 0$$

IN-TEXT QUESTIONS

1. Evaluate the following limits

(a) $\lim_{y\to 0^+}\frac{y+|y|}{y}$

(b) $\lim_{y\to\infty}\frac{x-3}{x^2+1}$

2. Find the asymptote of : $\dfrac{2y^3 - 3y^2 + 3y - 6}{y^2 + 1}$

3. Let $f(x) = x^2$, $g(x) = 1/x$. Determine $\lim\limits_{x\to\infty}$ for the following:

$f(x)\cdot g(x)$ (ii) $F(x)+g(x)$ (iii) $F(x)/g(x)$

ANSWER TO IN-TEXT QUESTIONS

1. (a) 2 (b) Limit is not defined

2. $x = 2y - 3$

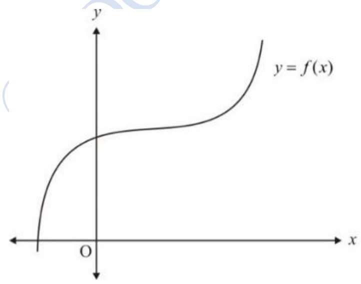
3. (i) ∞ $F(x) = \infty$ as $x \to \infty$

   (ii) ∞ $G(x) = 0$ as $x \to \infty$

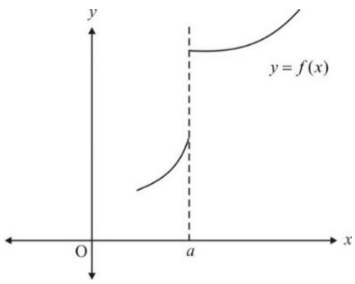
   (iii) ∞

7.5 CONTINUITY OF A FUNCTION

Continuity describes changes that occur over a period rather than suddenly. Thus, a function $Y = f(x)$ is said to be continuous if changes in independent variable (x) brings about changes in the function value (y). Geometrically, a function is continuous if its graph is connected, and it has no breaks. However, if there exists a break in the graph, then function is said to be discontinuous. This suggests that the value of the function that passes through a point, there is a sudden change in the value of the function. Graphically, both cases can be depicted as:



Continuous Function (a)



Discontinuous function (b)

In the graph above, there is a smooth line in the case of continuous function (a), while there is a break in the graph in discontinuous case (b).

7.5.1 Continuous Function

A function $f(x)$ is said to be continuous at a point 'h if the graph of $f(x)$ has no break at point 'a'. Thus, if $f(x)$ has a limit as x tends to a in its domain, and $f(x)$ tends to $f(a)$ when $x = a$, then function is said to be continuous at 'a'.

$f(a)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$ (7.5)

However, for $f(x)$ to be continuous at $x = a$, it must satisfy the following conditions:

(i) $F(x)$ must be defined at $x = a$

(ii) As $x \to a$ limit of $f(x)$ must exist

(iii) Limit of $f(x)$ must equal to $f(a)$

$$\lim_{x \to a} f(x) = f(a)$$

Unless all the above conditions are satisfied, $f(x)$ will be discontinuous at $x = a$.

7.5.2 Discontinuous Functions

If a function $f(x)$ does not satisfy all the conditions of continuity, then the function will be discontinuous. In general, there are two types of discontinuity that can arise:

(i) **Irremovable discontinuity**: If there exist no limit of the function $f(x)$ when x tends to a, then irremovable discontinuity arises.

(ii) **Removable discontinuity**: If there exist, limit of function $f(x)$ as $x \to a$ and is equal to K, but $\lim_{x \to a} f(x) \neq f(a)$, then $f(x)$ is discontinuous at $x = a$. This is considered as removable discontinuity, as $f(a)$ can be simplified and refined as K.

Example: Discuss the continuity of the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ at x = 2.

Solution: Given $x = 2 \Rightarrow f(x) = \frac{4-4}{2-2} = \frac{0}{0}$ is not defined. The function becomes discontinuous, having removable discontinuity. However, further simplification yields.

$$f(x) = \frac{(x-2(x+2)}{(x-2)} = x+2$$

Thus, $f(2) = 2+2 = 4$ and $\lim_{x \to 2} x+2 = 2$. Thus, limit of function exists, and function is defined. The first two conditions for a function to be continuous are satisfied.

The last condition states that limit of $f(x)$ must equal to $f(2)$.

Now, $\lim_{x \to 2} f(x) = \frac{0}{0}$ an indeterminate form

Thus, function $f(x)$ is undefined at $x = 2$.

Example: Determine at what value of x, the given function is continuous.

$$f(x) = \frac{x+4}{(x+1)(x+2)}$$

Solution: The function will be continuous for all values of x, except the case when, $(x+1)(x+2) = 0$. Thus, $f(x)$ will be continuous for all x except $x = -1$ and $x = -2$.

7.5.3 Properties of Continuous Function

Using the limits rules that we established in the previous section; we apply these rules in the case of continuous functions.

Let $f(x)$ and $g(x)$ be two continuous functions at h, then

(i) $f(x) \cdot g(x)$ will be continuous at h.

(ii) $f(x)+g(x)$ and $f(x)-g(x)$ are continuous at h.

(iii) $\frac{f(x)}{g(x)}$ will be continuous at h if $g(h)\neq 0$

(iv) $[f(x)]^{a/b}$ is continuous at 'h' if $[f(h)]^{a/b}$ is defined.

Apart from these properties, if we take a **composite function** i.e., a function where two functions are combined to form a new function. Suppose $f(x)$ and $g(x)$ are two continuous functions and let $P(x)$ be a composite function, then

$$P(x)=F(g(x))$$

Then, going by the above stated properties, if $g(x)$ is continuous at x = k and $f(\cdot)$ is continuous at $g(k)$, then $f(g(x))$ will be also continuous at x = k.

In general terms

"*Any function that can be constructed from continuous functions by combining one or more operations of addition, subtraction, division or multiplication (except zero), and composition is continuous at all points where it is defined*."

Sydsater & Hammond, 2009

Example: For what values of p, the following function is continuous for all x?

$$F(x)=\begin{cases} px-1 & \text{for } x\leq 1 \\ 3x^2+1 & \text{for } x>1 \end{cases}$$

Solution: The function $f(x)$ is continuous for all $x\neq 1$. If $x=1$ then $f(1)=p-1$. If x is greater than 1, then $f(x)=3x^2+1$ will be close to 4 and $f(x)\rightarrow 4$ as $x\rightarrow 1^+$.

For attaining continuity at x =1, requires that

$$f(1) = p - 1 = 4 \Rightarrow p = 5.$$

Thus, $f(x)$ is continuous for all x, even at $x = 1$ given $p = 5$.

7.6 ONE-SIDED CONTINUITY

Similar to the one-sided limits discussed in the section, we also have one-sided continuity. If a function $f(x)$ is defined on a domain consisting of open interval such as $(a, b]$ and $f(x) \to f(b)$ as $x \to b^-$, then $f(x)$ is **left continuous** at b.

On the other hand, $f(x)$ is defined on a domain $[b, d)$ and $f(x) \to f(b)$ as $x \to b^+$ then $f(x)$ is **right continuous** at b.

However, previously stated $f(x)$ is continuous at *b* if and only if $f(x)$ is continuous both from right and left side.

Sometimes, the domain of the function is defined on closed interval. Let say $f(x)$ is defined on closed interval [c, d], then $f(x)$ is said to be continuous in the interval [c, d], if it is continuous at $x = c$, at $x = d$ and at any point between c and d.

IN-TEXT QUESTIONS

1. Find the values of y for which the function is continuous.

(i) $f(y) = \dfrac{1}{\sqrt{2-y}}$ (iii) $|f| + \dfrac{1}{|y|}$

(ii) $f(y) = \dfrac{y^8 - 3y^2 + 1}{y^2 + 2y - 2}$

2. For what values of 'p' the function is continuous everywhere?

$$g(x) = \begin{cases} px^2 + 42 - 1 & x \le 1 \\ -x + 3 & x > 1 \end{cases}$$

ANSWER TO IN-TEXT QUESTIONS

1. (i) Continuous for all $y < 2$

   (ii) Continuous for all y, where $y \neq \sqrt{3}-1$ and $y \neq -\sqrt{3}-1$

   (iii) Continuous for all $y \neq 0$.

2. $p = -1$, the function is continuous.

7.7 SEQUENCES

We have often seen that economic data is represented in the form of sequence of numbers. Let say the figures for Gross Domestic Product (GDP0 for India from 2010 to 2021 is

$$a_1, a_2, a_3, a_4, \ldots. \qquad (7.6)$$

where, a_1 denotes GDP in the year 2010, a_2 denotes GDP in year 2011 and so on. In mathematics, the term 'sequences' is mostly understood as 'infinite sequencers' that is going on forever. However, the 'sequences' represents a function whose domain is the set of real numbers. For ex: the sequence of odd numbers; 1, 3, 5, 7, ….

To be more precise, let N be set of all natural numbers and if n $\in$N, then a sequence of real numbers will be a f unction from N to R. if 'p' is such a function then,

$$\{p_n\} = p_1, p_2, p_3, \ldots. p_n \text{ is called}$$

the sequence and elements p_1, p_2, p_3 are called terms of sequence.

Example; If $a_n = 4n$ for n = 1, 2, 3, 4, ….

It will give the sequence as 4(1), 4(2), 4(3, 4(4), ….

$\Rightarrow$ 4, 8, 16, 24, ….

Example: Let $b_n = 30 - 4n$ for n = 1, 2, 3….

then we will obtain the sequence as 30 –(1), 30 – 9(2), 30 – 9(3)….

$\Rightarrow$ 21, 12, 3, …...

7.7.1 Infinite Sequences

It supposes, we defined $\{a_n\} = \frac{1}{2n}$ with n = 1, 2, 3, …., then we get sequence of the form

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots, \frac{1}{2n}, \ldots$$

then, this sequence is known as 'infinite sequence', where a_n infinite sequence is a function, whose domain is the set of positive integers.

7.7.2 Convergence of Sequence

From the previous example, if we take the value of $n = \infty$, then the value of last term becomes zero, that is sequence will converge to zero. By convergence, we are basically examining the fact that whether the subsequent terms of the series is getting closer to a **value** as '*n*' increase.

If the sequence approaches a definite value, it is said to **converge** to that value. More formally, if sequence $\{a_n\}$ converges to a number a', if a_n gets closer to number '*a*' for '*n*' being sufficiently large.

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad \underset{n\to\infty}{\text{lt}}\ a_n = a \quad \text{or} \quad a_n \to a \text{ when } n \to \infty.$$

If sequence $\{a_n\}$ does not converge to any definite value, then it is said to **diverge**. Let suppose a_n = 100^n as n tends to ∞, a_n will also tend to ∞.

Example: Determine whether the following sequences converges or not when $n \to \infty$.

(a) $a_n = \frac{n+1}{2n}$ (b) $a_n = \frac{n^4+1}{n^3+2}$

Solution:

(a) $a_n = \frac{1}{2}\left[1+\frac{1}{n}\right]$ as $n \to \infty$ then $\frac{1}{n} \to 0$

Thus $\lim_{n\to\infty} a_n = \frac{1}{2}$. Thus, it is convergent.

(b) $a_n = \frac{n+\frac{1}{n^3}}{1+\frac{2}{n^3}}$ (Taking n^3 common from both the numerator and denominator)

As $n\to\infty$, $1/n \to 0$ but $n \to \infty$.

Thus, $\lim_{n\to\infty} a_n = \infty$. Thus, it is divergent.

IN-TEXT QUESTION

1. Examine whether the sequence converges or diverges

(a) $a_n = \frac{n^3-1}{n^2}$ (b) $a_n = 1+\frac{1}{4}n$ (c) $a_n = 4-\frac{2}{n}$

2. Let $\mu_n = \frac{n+1}{2n}$ and $\alpha_n = \frac{n^2+2n+1}{3n^2+1}$

Determine

(a) $\lim_{n\to\infty} \mu_n$ (c) $\lim_{n\to\infty} \mu_n \alpha_n$

(b) $\lim_{n\to\infty} \alpha_n$ (d) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}$

ANSWER TO IN-TEXT QUESTIONS

1.	(a) Diverges	(b) Diverges	(c) Converges	
2.	(a) 1/2	(b) 1/3	(c) 1/6	(d) 3/2

7.8 SERIES

In this section, we will learn about finite and infinite geometric series. A series is generated by a sequence { a_n } which is a summation of first n terms of a sequence such that s_n ,

$$s_n = \sum_{r=1}^{n} a_r$$

Example: Let $\{a_r\} = \dfrac{n+1}{2}$ with n = 1, 2, 3, 4. Find s_n

Solution: $\{a_r\} = \left(\dfrac{1+1}{2}\right) + \left(\dfrac{2+1}{2}\right) + \left(\dfrac{3+1}{2}\right) + \left(\dfrac{4+1}{2}\right) = 7$

7.8.1 Finite Geometric Series

Let us consider a sequence of 'n' numbers such that

$$a, at, at^2, at^3, ... at^{n-1}$$

where each term is obtained by multiplying the previous term by a constant 't'. Then, the summation of these sequences is:

$$\boxed{s_n = a + at + at^2 + at^3 + ... + at^{n-1}} \qquad (7.6)$$

This sum is known as finite geometric series with quotient 't'. To find the sum of the series, let us multiply both sides of (7.6) by the constant 't', to get:

$$ts_n = at + at^2 + at^3 + at^4 + \ldots + at^{n-1} + at^n \qquad (7.7)$$

Now, if we subtract (7.7) from (7.6)

$$s_n - ts_n = a - at^n \qquad (7.8)$$

Assume, $t = 1$ then in that case from (7.6)

$$s_n = a_n$$

and if $t \neq 1$, then from (7.8),

$$(1-t)s_n = a - at^n$$

$$\boxed{s_n = \frac{a - at^n}{(1-t)}}$$

Thus, the summation formulae for a finite geometric series is

$$a + at^2 + at^3 + at^4 + \ldots + at^{n-1} = a\left[\frac{1-t^n}{1-t}\right], \quad t \neq 1 \qquad (7.9)$$

7.8.2 Infinite Geometric Series

From the previous equation (7.6), if the geometric series is infinite such that

$$a + at^2 + at^3 + \ldots + at^{n-1} + \ldots. \qquad (7.10)$$

If in this case n tends to ∞, $n \to \infty$, we know s_n of first n terms of (7.10),

$$s_n = a\left[\frac{1-t^n}{1-t}\right], (t \neq 1) \qquad (7.11)$$

Here only t^n depends on 'n', then as $n \to \infty$,

(i) $t^n \to 0$ if $-1 < t < 1$ and $s_n = \frac{a}{1-t}$

(ii) If t > 1 or t ≤ –1 then t^n does not tend to any limit.

Example: Let t = 1/5, then $(1/5)^n$ as n → ∞, t → 0. Here the value of 't' lies between –1 < t < 1. If t = 5 then $(5)^\infty \to \infty$ then their exist no value.

7.8.3 Convergence of Series

Let $\{a_n\}$ be the sequence and $\{s_n\}$ be the series generated from it. If s_n approaches to a limit s as $n \to \infty$, we say that series { s_n } is **convergent**. In our case,

$$s_n = a\left[\frac{1-t^n}{1-t}\right] \qquad (7.12)$$

If $-1 < t < 1$, then $t^n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} s_n = a\left[\frac{1-0}{1-t}\right] = \frac{a}{1-t} \qquad (7.13)$$

If $|t| \geq 1$, then the series is **divergent.**

IN-TEXT QUESTIONS

1. Does the sequence

$$u_n = \frac{3n^2}{n^3+3} \text{ tends to a limit?}$$

2. Find the sum of finite geometric series

$$1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+....+\frac{1}{5^{n-1}}$$

In $n \to \infty$, what limit does s_n tends to?

3. Examine whether the following series converges or diverge

(i) $s_n = \dfrac{5+8n^2}{2-7n^2}$ (ii) $s_n = \dfrac{n^2}{5+2n}$

(iii) $\sum_{n=1}^{\infty} \left(-\dfrac{1}{4}\right)^n$

4. Find the sum of the series

$$\sum_{n=1}^{\infty} a\left(1+\frac{k}{100}\right)^{-n} \qquad k > 0$$

ANSWER TO IN-TEXT QUESTIONS

1. Diverges

2. $s_n = \dfrac{5}{2}[1-(1/5)^n]$ as $n \to \infty$, $\sum_{n=1} \dfrac{1}{5^{n-1}} = 5/2$

3. (i) 8/7 converges (ii) Diverges

(iii) Geometric series with quotient – ¼ converges to 1/5.

4. It has quotient $\left(1+\dfrac{k}{100}\right)^{-1}$ and sum.

$$\frac{a}{\left[1-\left(1+\dfrac{k}{100}\right)^{-1}\right]} = a\left(1+\frac{100}{k}\right)$$

7.9 APPLICATIONS OF SEQUENCES AND SERIES

The concept of sequences can be used in economic applications to determine the present value' of a sum of money to be received in future at some point in time. Suppose an individual wants to invest an amount of. 100 at an annual interest rate of 10%. Then the amount received at the end of the year is equivalent to.100

(1+ 10/100) = ₹100. This corresponds to saying that the present value of amount A to be received in one year's time is $y = \dfrac{A}{(1+t)}$, where r is the rate of interest (rate of return). In our example, ₹100 is the 'present discounted value' of ₹110 and 10% is the discount rate, while ₹100/₹110 is the discount factor (1/(1 + r)).

Let us suppose, an individual has to make four annual payments, such that it receives ₹100 after the 1st year, ₹150 in the second, `200 in third and ₹ 250 in fourth year. We need to determine the amount that must be invested today with a given interest rate of 11%. In other words, we need to ascertain the present value of these four payments. Thus, in year 1, to receive ₹100, he must deposit ₹A_1 such that:

$$₹\, A_1\left(1+\frac{11}{100}\right) = 100 \Rightarrow A_1 = \frac{100}{1.11}$$

Similarly, for year 2, 3 and 4, we have

$$A_2\left(1+\frac{11}{100}\right)^2 = 150 \Rightarrow A_2 = \frac{150}{(1.11)^2}$$

$$A_3\left(1+\frac{11}{100}\right)^3 = 200 \Rightarrow A_3 = \frac{200}{(1.11)^3}$$

and $$A_4\left(1+\frac{11}{100}\right)^4 = 250 \Rightarrow A_4 = \frac{250}{(1.11)^4}$$

Thus, the total present value of the four annual payments is amount A that must be deposited today

$$A = A_1 + A_2 + A_3 + A_4 = \frac{100}{(1.11)} + \frac{150}{(1.11)^2} + \frac{200}{(1.11)^3} + \frac{250}{(1.11)^4}$$

This will equal to 90.09 + 121. 743 + 146.238 + 374.5 = 737.57.

In general, if *n* successive payments must be made with A_1 after year 1, A_2 after two years, A_3 after 3 years and so on, then the present value of all these payments with an interest rate of r% per year would be:

$$P_n = \frac{A_1}{(1+r/100)} + \frac{A_2}{(1+r/100)^2} + \ldots + \frac{A_n}{(1+r/100)^n}$$

or equivalently

$$\boxed{P_n = \sum_{i=1}^{n} \frac{A_i}{(1+r/100)^i}} \qquad (7.14)$$

Using the concept of finite geometric series, with first term as $\frac{A}{(1+r/100)}$ and

t= $\frac{1}{(1+r/100)}$, then

$$A_n = \frac{a}{(1+r/100)}\left[\frac{1-(1+r/100)^{-n}}{1-(1+r/100)^{-1}}\right]$$

Let r/100 = p, then

$$A_n = \frac{a}{(1+p)}\left[\frac{1-(1+p)^{-n}}{1-(1+p)^{-1}}\right] = \boxed{\frac{a}{p}\left[1-\frac{1}{(1+p)^n}\right]} \qquad (7.15)$$

Thus, the present value of 'n' installments with r% rate of interest is given by $\frac{a}{p}\left[1-\frac{1}{(1+p)^n}\right]$ with $p = r/100$.

In a special case, if $p > 0$ but $n \to \infty$, then

$$\lim_{n\to\infty} \frac{a}{p}\left[1-\frac{1}{(1+p)^n}\right] = \frac{a}{p} \qquad (7.16)$$

Example: Determine the present value of 15 annual deposits of ₹50,000 if the first payment has to be made one year from now with interest rate of 8% per year?

Solution: Here, $A_{15} = \frac{50000}{0.08}\left[1-\frac{1}{(1.08)^{15}}\right]$

$\Rightarrow \qquad A_{15} = 50,000 \times 8.55948$

= ₹ 4, 27,974.

INTERNAL RATE OF RETURN

Internal rate of return is the discount rate used to estimate the profitability of investments. It is defined as an interest rate that makes the present value of all the payment equal to zero. Thus, if the investment projects for 'n' time period give the returns $I_0, I_1, I_2, ...I_{n-1}$, then the internal rate of return is 'α' such that, I_0 being initial investment:

$$I_0 + \frac{I_1}{1+\alpha} + \frac{I_2}{(1+\alpha)^2} + \frac{I_3}{(1+\alpha)^3} + ... + \frac{I_{n-1}}{(1+\alpha)^{n-1}} = 0 \qquad (7.17)$$

If there is a choice between two investments with a different internal rate of return, then the investment that has a higher rate of internal return should be preferred.

IN-TEXT QUESTIONS

1. What is the present value of 10 annual deposits of ₹ 1000 each with first deposit is made 1 year from new at interest rate of 14% per year?
2. A firm wants to invest in machinery with three payment options:
   (a) Pay ₹ 67,000 in cash
   (b) ₹12,000 per year for 8 years, where the first installment has to be paid once.
   (c) ₹ 22000 in cash payment now ₹27000 per year for 12 years with first installment to be paid after one year.

   If a firm has ₹67000 cash available and interest rate is 11.9%, determine which option is least expensive?

ANSWER TO IN-TEXT QUESTIONS

1. ₹5216.12
2. Option you have to pay ₹67000

   Option b has present value $\frac{12000(1+0.115)}{0.115}(1-(1.115)^{-8})$

   = ₹67644.42

   While option c = $22000+7000\frac{(1.115)^{12}-1}{0.115(1.115)^{12}}$ = ₹66,384.08

   Thus option 'c' is the least expensive

7.10 TERMINAL QUESTIONS

Determine the values of real number m and n such that the function y=f(x) is continuous.

(a) f(x)= $\begin{cases} x+m^2 & if\ x \leq 2 \\ x-n^2 & if\ x > 2 \end{cases}$

(b) $f(x)=\begin{cases} mx^3 & if\ x \le 2 \\ nx^2 & if\ x > 2 \end{cases}$

1. Let

$$G(x) = \begin{cases} x^2\ if\ x < 1 \\ 3 - 2x\ if\ 1 \le x \le 2 \\ x^3 - 8\ if\ x > 2 \end{cases}$$

Is the function continuous at x=0 and x=1?

2. Given the infinite series

$1+\frac{2x}{3} + (\frac{2x}{3})^2 + (\frac{2x}{3})^3 + - - - - - - -$

For what values of x does the series converge? Find the sum of the series if x=1.2.

Answers:

1. (a) m=n=0

   b) Any value of m and n such that n=2m

2. Yes, continuous at both x= 0 and 1.

3. x ϵ (-3/2, 3/2) and series will converge to value $\frac{3}{3-2x}$. For x=1.2 its value is 5.

7.11 SUMMARY

The unit introduced the concept of 'limit' which is associated with approaching to a value. We discussed the concepts of right-hand and left-hand limits and the properties of limits. After limits, we extend discussion to continuity. A function was said to be continuous, it there is no break in the graph. The unit focuses on properties of continuous functions and different types of discontinuity.

After that, we introduced a function called sequences. From these sequences, we derived the series. A sequence represents a mapping of numbers (natural numbers) to the set of elements. Subsequently, the unit discussed the concept of series, which was obtained by adding the terms of the sequence. The concept of convergence and divergence of both series and sequence were also analyzed.

Lastly, the unit discussed the applications of sequences and series in economic applications such as calculation of present discounted value and internal rate of return.

7.12 REFERENCES

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LESSON 8

SINGLE VARIABLE DIFFERENTIATION

STRUCTURE

8.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to :

1. Discuss the geometric interpretation of the derivative of a function.
2. Obtain the derivative of the sums, products and quotients of functions.
3. Apply the generalized power rule and chain rule to find out derivatives and
4. Identify the applications of derivatives in economics.

8.2 INTRODUCTION

A key question in most disciplines, including economics, is how quickly the value of a variable changes over time. In other words, we are often interested in finding out rates of change of variables. In economics, the rate of change is an essential component of comparative statistics, which compares different equilibrium states of a variable. For instance, given an initial equilibrium level of income Y^*, we might be interested in finding out the change in this income level due to an increase in an exogenous variable such as the amount of government expenditure.

Mathematically, the rate of change of a function (of one or more variables) is described by finding its derivative, a key concept in differential calculus developed by Isaac Newton and Gottfried Leibniz. In this lesson, we will first discuss the geometric interpretation of the derivative of a function, followed by some rules for calculating derivatives of functions of different types. The lesson will also take a deep dive into the economic applications of derivatives.

8.3 GEOMETRIC INTERPRETATION OF THE DERIVATIVE OF A FUNCTION

Geometrically, the derivative of a function at a given point is nothing but the slope of the tangent to the graph of the function at the point. For example, consider the graph of a function shown in $Figure$ 1.

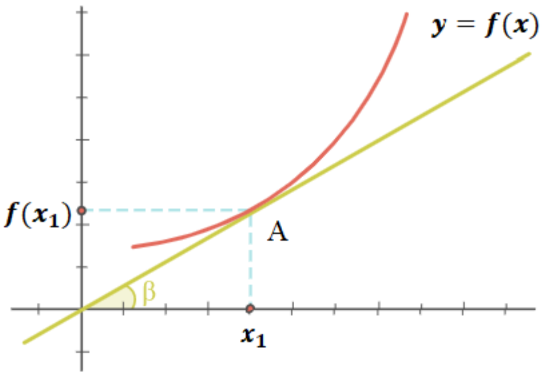

Figure 1

In Figure 1, the derivative of the function at point A is given by the slope of the tangent to the graph at A and we denote this number by $f'(x_1)$ (read as f dash x_1 or f prime x_1).

Let us now discuss in detail what we mean by the tangent to a curve at a point. For this, let us consider point A a fixed point on the curve and let B be a nearby point on the curve.

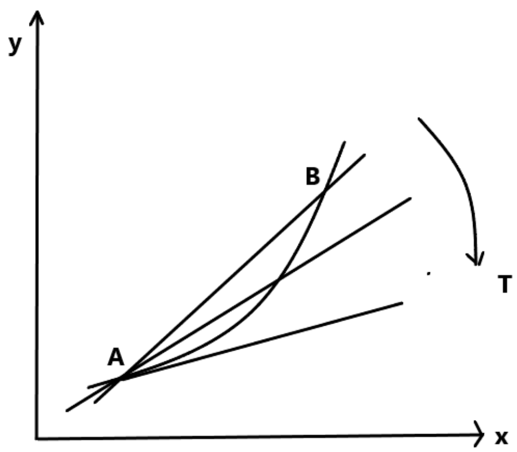

Figure 2

The straight line passing through A and B is called a Secant. If we let B move along the curve towards T, the secant will rotate around B. The limiting straight-line AT towards which the secant tends is called the tangent to the curve at A.

We now wish to find out the slope of this tangent to the curve at point A. Consider Figure 3. Here, the coordinates of point A are $\left(a, f(a)\right)$.

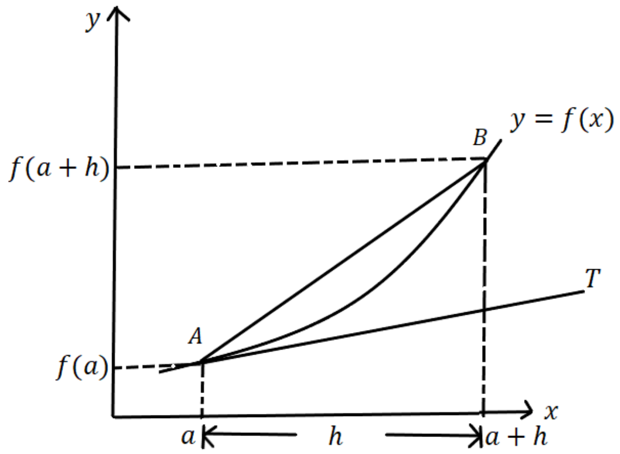

Figure 3

Point B is another point on the curve which lies close to point A. The Coordinates of point B are $\left(a+h, f(a+h)\right)$ where h is a small number $\neq 0$.The slope of secant AB is then given by

$$m_{AB} = \frac{\left(f(a+h) - f(a)\right)}{(a+h) - (a)}$$

$$m_{AB} = \frac{f(a+h) - f(a)}{h}$$

This fraction is often called Newton (or Differential) quotient of f.

Figure 3 shows that as h tends to 0, point B tends towards point A. Hence, the slope of the tangent to the curve at point A is the number that m_{AB} approaches as h tends to 0.

Hence, the derivative of the function $y = f(x)\ at\ x = a$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad (1)$$

Once we have found $f'(a)$, it is easy to find the equation for the tangent at $(a, f(a))$.

The same is given by:

$$y - f(a) = f'(a)\,(x - a)$$

8.3.1 Other Notations for Derivatives

A general notation for the derivative of a function $y = f(x)$ is $f'(x)$. We can also denote the same by y' (y prime or y dash).

The other notation is the differential notation. Note that the slope of the secant AB in figure 3 was given by

$$m_{AB} = \frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{\Delta y}{\Delta x} \left[\frac{Change\ in\ value\ of\ y}{Change\ in\ value\ of\ x}\right]$$

Since $$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

The term $\frac{dy}{dx}$ or dy/dx is the differential notation of the derivative of a function. Since $y = f(x)$, $\frac{dy}{dx}$ can also be written as $\frac{d(f(x))}{dx}$ or $\frac{d}{dx}f(x)$.

Letting $a = x$ in equation (1), we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \frac{dy}{dx} = y'$$

If this limit exists, we say that the function is differentiable at x. The process of finding the derivative of a function is called differentiation.

IN-TEXT QUESTIONS

1. Given $f(x) = 4x^2 + 20$. Then, $f'(a)$ is given by:

$(a)\ \lim_{h\to 0}\frac{4(a+h)^2 + 20 - (4a^2 + 20)}{h}$

$(b)\ \lim_{h\to 0}\frac{4(a+h)^2 - 20 - (4a^2 + 20)}{h}$

$(c)\ \lim_{h\to 0}\frac{4(a+h)^2 - (4a^2 + 20)}{h}$

$(d)\ \lim_{h\to 0}\frac{4(a+h)^2 - (4a^2 + 20)}{h + 20}$

8.4 RULES FOR DIFFERENTIATION

- **CONSTANT-FUNCTION RULE**

If $y = f(x)$ is a constant function, for example, $y = k\ \forall\ x$, then the derivative $f'(x) = 0$.

Geometrically, the graph of $y = k$ is a straight line parallel to the x axis. Hence, the tangent to the graph has a slope of zero at each point.

Alternatively, using the definition of derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

If $f(x) = k$,

$$f'(x) = \lim_{h \to 0} \frac{k - k}{h} = 0$$

It is noteworthy here to distinguish between $f'(x) = 0$ and $f'(a) = 0$. $f'(x) = 0$ means that the derivative of the function is 0 for all values of x while $f'(a) = 0$ means that the derivative of the function is 0 at $x = a$.

We now state the remaining results without proof.

- **POWER FUNCTION RULE**

If $f(x) = x^n$ where n is any arbitrary constant

Then,

$$f'(x) = nx^{n-1}$$

In differential notation,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

GENERALIZED POWER FUNCTION RULE

Let,

$$y = [f(x)]^n$$

Then,

$$y' = n[f(x)]^{n-1} . f'(x)$$

- **DIFFERENTIATION OF SUMS AND DIFFERENCES**
  a) When we take the sum of two functions:

$$F(x) = f(x) + g(x)$$

$$F'(x) = f'(x) + g'(x)$$

  b) When we take the difference of two functions:

$$F(x) = f(x) - g(x)$$

$$F'(x) = f'(x) - g'(x)$$

- **PRODUCT RULE**

$$F(x) = f(x).g(x)$$

$$F'(x) = f(x)g'(x) + f'(x)g(x)$$

- **QUOTIENT RULE**

$$F(x) = \frac{f(x)}{g(x)}$$

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

- **CHAIN RULE**

$$F(x) = f(g(x))$$

Here, $F(x)$ is a composite function, $g(x)$ is known as its kernel and f as the exterior function.

Then,

$$F'(x) = f'\big(g(x)\big).g'(x)$$

EXAMPLE 1: Differentiate the following functions with respect to x.

a) $\frac{9x-10}{2x-4}$

b) $\sqrt{x^2 + 10}$

c) $v = 6t^2\ , where\ t = 6x + 15$

SOLUTION

a) $f(x) = \frac{9x-10}{2x-4}$

Using the quotient rule,

$$f'(x) = \frac{9(2x-4) - (9x-10)(2)}{(2x-4)^2}$$

$$= \frac{18x - 36 - 18x + 20}{(2x-4)^2}$$

$$= \frac{-16}{[2(x-2)]^2}$$

$$= \frac{-4}{(x-2)^2}$$

b) $f(x) = \sqrt{x^2+10} = (x^2+10)^{1/2}$

$$f'(x) = \frac{1}{2}(x^2+10)^{\frac{1}{2}-1}.(2x)$$

$$= \frac{1}{2}(x^2+10)^{-\frac{1}{2}}.(2x)$$

$$= \frac{x}{\sqrt{x^2+10}}$$

(c) $v = 6t^2 \ where, t = 6x + 15$

This is of the form

$$F(x) = f(g(x))$$

were,

$$f = 6t^2 \ and \ g(x) = 6x + 15$$

Hence,

$$F'(x) = f'\big(g(x)\big).g'(x)$$

$$f'\big(g(x)\big) = 12t = 12(6x+15)$$

$$g'(x) = 6$$

$$\therefore F'(x) = 72(6x+15)$$

IN-TEXT QUESTIONS

2. Find the derivative of the following functions:

a) $\frac{x^{1/2}-20}{x^{1/2}+10}$

b) $(t^2+1)\sqrt{t}$

c) $x^n(a\sqrt{x}+20)$

d) $(x^4-x^2)(5x^3+2x^2)$

8.5 HIGHER-ORDER DERIVATIVES

If $y = f(x)$, then $y' = \frac{dy}{dx} = f'(x)$ is called the first derivative of $f(x)$. If $f'(x)$ is also differentiable, then we can find further higher-order derivatives of $f(x)$.

- **SECOND-ORDER DERIVATIVE**

$$y'' = f''(x) = \frac{d^2y}{dx^2} = y^{(2)} = f^{(2)}(x)$$

The number 2 in the parentheses denotes that we are referring to the order of the derivative here.

Similarly,

- **THIRD-ORDER DERIVATIVE**

$$y''' = f'''(x) = \frac{d^3y}{dx^3} = y^{(3)} = f^{(3)}(x)$$

- $\boldsymbol{n^{th}}$**ORDER DERIVATIVE**

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

EXAMPLE 3: Compute y''' for $y = 130x - \frac{1}{3}x^3$

SOLUTION:

$$y = 130x - \left(\frac{1}{3}\right)x^3$$
$$y' = 130 - \frac{1}{3}.3x^2$$
$$= 130 - x^2$$
$$y'' = -2x$$
$$= y''' = -2$$

IN-TEXT QUESTIONS

3. $f^{(4)}(1)$ for $y = 500x^{-4}$ is:

(a) 2,50,000

(b) 3,00,000

(c) 3,70,000

(d) 4,20,000

8.6 APPLICATIONS OF DERIVATIVES IN ECONOMICS

In economics, we frequently encounter several "marginal" concepts such as marginal utility, marginal cost, marginal revenue etc. In general, a marginal function is defined as the change in the total function due to a unit change in the independent variable.

For example, marginal cost is the cost incurred when an additional unit of output is produced. Similarly, marginal revenue is the addition to total revenue when an extra unit of output is sold.

When the underlying total functions are continuous and differentiable, the marginal functions can be obtained by taking the first-order derivative of the total function.

To understand this, let us take the example of Marginal revenue (MR). Now by definition,

$MR(x) = TR(x + 1) - TR(x)$, that is the additional revenue obtained by selling one more unit of x.

By the definition of derivative,

$$TR'(x) = \lim_{h \to 0} \frac{TR(x+h) - TR(x)}{h}$$

Since a firm would sell many units of output (x), we can consider h=1 as a number close to 0.

Hence,

$$TR'(x) = TR(x+1) - TR(x) = MR(x)$$

The other marginal functions can similarly be obtained as the derivative of the corresponding total functions.

For example, marginal cost is the derivative of the total cost function with respect to output. Marginal utility is the derivative of the total utility function.

EXAMPLE 4: Suppose the inverse demand function is given by

$$p = \frac{100}{q+5}$$

Show that the marginal revenue is always positive. Also, show that as output increases, total revenue increases while marginal revenue decreases.

SOLUTION:

$$TR = p.q = \frac{100q}{q+5}$$

$MR = \frac{dTR}{dq} = \frac{100(q+5) - 100q}{(q+5)^2} = \frac{500}{(q+5)^2}$ which is positive for all values of q.

Hence, marginal revenue is positive at all levels of output. Since marginal revenue is the slope of the total revenue, the positive value of the marginal revenue implies that as output increases, total revenue increases.

The slope of the MR curve is given by:

$\frac{dMR}{dq} = \frac{d}{dq}\left(\frac{500}{(q+5)^2}\right) = \frac{0(q+5)^2 - 500(2)(q+5)}{(q+5)^4} = \frac{-1{,}000}{(q+5)^3} < 0$ for all q>0.

Hence, as output increases, marginal revenue falls.

EXAMPLE 5: Suppose the total cost function of a firm is given by

$TC = 0.02q^3 - 4q^2 + 800q.$

Find the firm's marginal cost and average cost functions. At what level of output is MC=AC?

SOLUTION:

$$MC = \frac{dTC}{dq} = 0.06q^2 - 8q + 800$$

$$AC = \frac{TC}{q} = \frac{0.02q^3 - 4q^2 + 800q}{q} = 0.02q^2 - 4q + 800$$

For AC = MC

$0.06q^2 - 8q + 800 = 0.02q^2 - 4q + 800$

$0.04q^2 = 4q$

$q(0.04q - 4) = 0$

Therefore, $q = 0 \; or \; q = \frac{4}{0.04} = 100$

IN-TEXT QUESTIONS

4. If a firm faces the following demand function for its output:

$$x = 20 - 2p$$

Then, the MR curve for the firm is:

(a) 20-x

(b) 15-x

(c) 10-2x

(d) 10-x

8.7 DERIVATIVES AND RATES OF CHANGE

Instead of interpreting the derivative of a function as the slope of the tangent to its graph at a particular point, we can also interpret it as a rate of change. Consider a function $y = f(x)$.

Suppose the value of x changes from $a\ to\ a + h$. Then, the value of the function will change $f(a)$ to $f(a + h)$.

In other words, the change in the functional value when x changes from a to $a + h$ is $f(a + h) - f(a)$

Then, the **average rate of change of $f(x)$** when x changes from a to $a + h$ is:

$$\frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

If we take the limit as $h \to 0$, then we get the derivative of $f(x)$ at x=a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f'(a)$ is also referred to as the Instantaneous Rate of Change i.e., the rate of change at a particular instant (and not the interval, as was the case in average rate of change).

In some situations, we are also interested in analyzing the proportion $\frac{f'(a)}{f(a)}$ which is also referred to as the proportional or relative rate of change.

<u>NOTE:</u> When the independent variable is time, we often use the dot notation for differentiation with respect to time like $\dot{x}$. Here, $\dot{x} = \frac{dx}{dt}$.

For example,

$$x(t) = 15t^3 + 12t^2 + 6t + 5$$

$$x'(t) = 45t^2 + 24t + 6$$

$$\dot{x} = 45t^2 + 24t + 6$$

8.8 SUMMARY

In this lesson, we discussed the concept of derivatives of a function. We first discussed the geometrical interpretation of a derivative followed by the calculus definition of a derivative. We also looked at some of the rules for calculating derivatives of functions of different types. We then discussed some of the economic applications of derivatives. Particularly, we learnt that in economics, derivatives are often used to find the marginal functions of given total functions. For example, the marginal cost of function is obtained by taking the first order derivative of the total cost function. Finally, we learnt that instead of interpreting the derivative of a function as the slope of the tangent to its graph at a particular point, we can also interpret it as a rate of change.

8.9 GLOSSARY

Average Rate of Change: It measures the average change in the value of a function in a given interval

Derivative: The derivative of a function at a given point is nothing but the slope of the tangent to the graph of the function at a point.

Differentiation: The process of finding the derivative of a function is called differentiation.

Instantaneous Rate of Change: The rate of change at a particular instant, which is the same as the derivative of the function at that point.

8.10 ANSWERS TO IN-TEXT QUESTIONS

1. (a)

2. (a) $\frac{15}{(\sqrt{x}+10)^2\sqrt{x}}$

   (b) $\frac{5t^2+1}{2\sqrt{t}}$

   (c) $\frac{ax^{n-\frac{1}{2}}}{2} + n\left(a\sqrt{x}+20\right)x^{n-1}$

   (d) $x^3(35x^3+12x^2-25x-8)$

3. (d)

4. (d)

8.11 TERMINAL QUESTIONS

Q1. Find the first-order derivative of each of the following:

i. $y = \frac{20+x^2}{20-x^2}$

ii. $y = \frac{2t^3 - 5t + 70}{t^2 + 20}$

iii. $y = \left(\frac{2x+15}{x-10}\right)^4$

iv. $y = \left(u^2 + \frac{1}{u^2}\right)^2$

v. $y = x\sqrt{x^3 - 5}$

Q2. Find $\frac{dy}{dx}$ when $y = -3(t+1)^5$ and $t = \frac{1}{3}x^3$.

Q3. A firm has a demand function,

$$p = 1200 - 9q$$

where p is the price and q is the output.

Further, its production function is given by

$$q = L^{1/3}$$

where L is the units of labour employed. Find the marginal revenue product of labour when the firm employs 8 units of labor.

Q4. Let $u(t)$ and $v(t)$ be positive differentiable functions of t. Find an expression for $\frac{\dot{x}}{x}$ where $x = \mu[u(t)]^a[v(t)]^b$, where μ, a and b are constants.

Q5. Given a function,

$$y = x^n$$

Prove that the n^{th} derivative of y is given by $n!$.

8.12 REFERENCES

- Sydsaeter, K., Hammond, P. (2002). *Mathematics for economic analysis*. Pearson Education.
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LESSON 9

FURTHER TOPICS IN DIFFERENTIATION

STRUCTURE

9.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to :

1. Understand the concept of implicit differentiation.
2. Explain the concept of differentials and the associated rules.
3. Find the linear, quadratic, and higher order approximations to functions about the given points and
4. Explain the concept of elasticities of functions and analyze their application in economics.

9.2 INTRODUCTION

In the last lesson, we introduced the concept of derivative of a function, both in terms of the slope of the tangent to the curve and rate of change. We also discussed some of the basic rules of differentiation. In this lesson, we will look at the differentiation of functions where the dependent variable is not explicitly expressed as a function of the independent variable. We will then discuss the concept of differentials, followed by the methods to approximate functions using linear, quadratic and higher-order approximations.

The lesson will conclude with a discussion on elasticities, a concept which economists frequently use to characterize changes in the independent variable, instead of derivatives.

9.3 IMPLICIT DIFFERENTIATION

Up until now, we have discussed how to differentiate functions of the type

$$y = f(x)$$

For example:

$$y = 3x^5 + 20, \qquad y = \frac{3x+5}{2x+20}$$

These are called as explicit functions because here the variable y has been explicitly expressed as a function of x.

However, if we write $y = 3x^5 + 20$ as $y - 3x^5 - 20 = 0$, we no longer have an explicit function but an equation that implicitly describes the function.

Now, if we wish to differentiate an implicit function, one way is to express y as a function of x and then apply the usual rules of differentiation.

However, in some cases, it may not be possible to explicitly express $y$as a function of x.

For example, in the case of an equation of the type $x^2 + y^3 = y^5 - x^2 + 6y$, it is not possible to express y as a function of x.

In such cases, we differentiate both the left-hand and right-hand sides of the equation with respect to x, considering y as a function of x. We then find the expression for dy/dx from the resulting equation.

EXAMPLE 1: Find $\frac{dy}{dx}$ if $6x^3 + 4x^2y + 5xy^2 + 2y^3 = 0$

SOLUTION

$$6x^3 + 4x^2y + 5xy^2 + 2y^3 = 0$$

Differentiating each term with respect to x

$$18x^2 + 8xy + 4x^2y' + 5y^2 + 10xyy' + 6y^2y' = 0$$

$$y'(4x^2 + 10xy + 6y^2) = -18x^2 - 8xy - 5y^2$$

$$y' = \frac{dy}{dx} = \frac{-(18x^2 + 8xy + 5y^2)}{4x^2 + 10xy + 6y^2}$$

EXAMPLE2: Consider the standard macroeconomic framework

$$Y = C + I$$

$$C = f(y)$$

where Y is income, C is consumption, and I is investment. Find an expression for $\frac{dY}{dI}$ and interpret it.

SOLUTION

$$Y = C + I$$

$$C = f(y)$$

$$\therefore Y = f(y) + I$$

Differentiating both sides of the equation with respect to I,

$$\frac{dY}{dI} = f'(y)\frac{dy}{dI} + 1$$

$$\frac{dY}{dI}\left(1 - f'(y)\right) = 1$$

$$\frac{dY}{dI} = \frac{1}{1 - f'(y)}$$

$\frac{dY}{dI}$ is the change in income due to a change in investment. $f'(y) = \frac{dC}{dY}$ is the change in consumption due to change in income, which is referred to as the marginal prosperity to consume (MPC) in economics. Generally, the value of MPC lies between 0 and 1.

Hence,

$$\frac{dY}{dI} = \frac{1}{1 - f'(y)}$$

will be greater than 1.

$\frac{dY}{dI}$ is also referred to as the *Investment Multiplier*.

IN-TEXT QUESTIONS

1. Suppose $MPC = 0.5$. Then, the investment multiplier is:

   (a) 10

   (b) 20

   (c) 30

   (d) 40

9.4 DIFFERENTIAL OF A FUNCTION

Consider a differentiable function $y = f(x)$. Suppose the value of x changes by a small amount dx to $x + dx$.

We know,

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)dx \qquad (1)$$

Here, $f'(x)dx$ is called the differential of $y = f(x)$ and is denoted by dy or df. Let us now depict graphically the change in the value of the function $y = f(x)$ when x changes to $x + dx$.

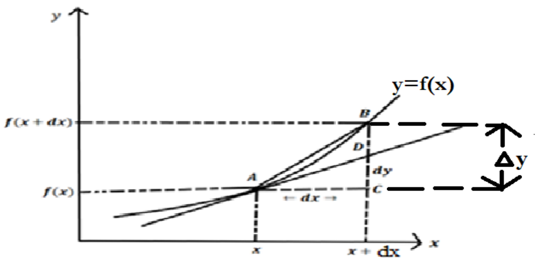

Figure 1

In Figure 1, as the value of x increase by a small amount dx, the functional value increases from $f(x)$ to $f(x+dx)$.

The change in the functional value is, therefore

$$\Delta y = f(x+dx) - f(x)$$

This is shown by the length of the line BC. Using differentials, it is possible to provide an approximate measure of this change. This approximate change is denoted by dy and is given by
$f'(x)dx$ (Using equation (1)). This is represented by line segment DC in the figure. Note that the gap between BC and DC i.e, between the actual change in $y(\Delta y)$ and approximate change in $y(dy)$ will be small for small values of dx.

9.4.1 Rules for Differentials

All the rules for differentiation discussed in the previous chapter can be expressed in terms of differentials as well. Hence, if $u = f(x)$ and $v = f(x)$ are two differentiable functions of x, then the following results hold:

(i) $d(au + bv) = adu + bdv$

(ii) $d(uv) = vdu + udv$

(iii) $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

PROOF OF (i) Since,

$$dy = f'(x)dx$$

$$d(au + bv) = (au + bv)'dx$$

$$= (au)'dx + (bv)'dx$$

$$= au'dx + bv'dx$$

$$= adu + bdv$$

The other results can also be proved in a similar fashion.

9.5 USE OF DIFFERENTIALS IN APPROXIMATION OF NON-LINEAR FUNCTIONS

9.5.1 Linear Approximation of a Function

Let us assume that $y = f(x)$ is a non-linear function as shown in Figure 2. We want to approximate this function when x is very close to an initial value, let us say a.

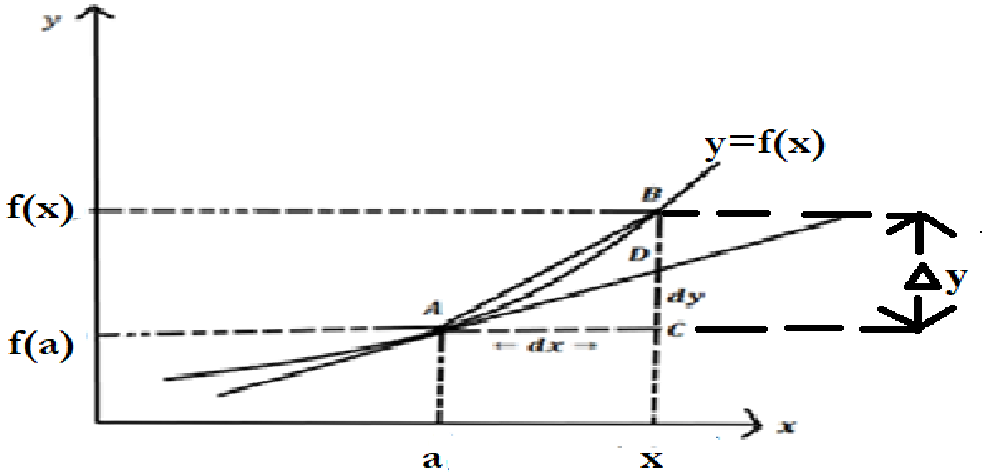

Figure 2

$$\Delta y = f(x) - f(a) \longrightarrow (2)$$

For small dx, $\Delta y \approx dy = f'(a)dx$

$$= f'(a)(x - a)$$

$$(\because dx = x - a)$$

From (2)

$$f(x) = f(a) + \Delta y$$

$$\approx dy + f(a)$$

$$= f'(a)(x - a) + f(a)$$

$$\therefore \; f(x) \approx f(a) + f'(a)(x - a)$$

EXAMPLE 3 Let $y = x^4 - 15$. Suppose the value of x changes from 2 to 1.99. Find the approximate change in y as well as the changed value of y.

SOLUTION We know, $f(x) - f(a) = \Delta y \rightarrow f(x) = f(a) + \Delta y$. If $\Delta y \approx dy \Rightarrow f(x) \approx f(a) + f'(a)(x - a)$.

Here, $a = 2$(the initial value of x)

Now,

$$f'(x) = 4x^3$$

$$f'(a) = f'(2) = 4(2)^3 = 32$$

Approximate change in y

$$f'(a)(\mathrm{x} - \mathrm{a})$$

$$= f'(2)(1.99 - 2)$$

$$= (32)(-0.01) = -0.32$$

The new value of y

$$f(1.99) \approx f(2) + f'(2)(1.99 - 2)$$

$$= (2^4 - 10) - 0.32 = 6 - 0.32 = 5.68$$

EXAMPLE 4 Find the linear approximation of the function $y = f(x) = \frac{1}{1+x}$ about a=0.

SOLUTION

$$f(x) \approx f(a) + f'(a)(x - a)$$

$$f(x) = \frac{1}{1+x} \;,\quad f'(x) = \frac{0(1+x) - 1}{(1+x)^2} = \frac{-1}{(1+x)^2}$$

$$f'(0) = -1$$

$$\therefore\; f(x) \approx f(0) + (-1)(x - 0) = \frac{1}{1+0} - x = 1 - x$$

9.5.2 Quadratic Approximation of A Function

Suppose we want to improve the accuracy of our approximation. In that case, we can approximate the function $y = f(x)$ by a quadratic polynomial when x is close to a.

The quadratic approximation is given by

$$f(x) \approx \alpha_0 + \alpha_1(x - a) + \alpha_2(x - 2)^2 \longrightarrow (3)$$

Here,

$$\alpha_0 = f(a)$$

To get α_1, we will differentiate equation (3) and put $x = a$

$$f'(x) = \alpha_1 + 2\alpha_2(x - a)$$

$$f'(a) = \alpha_1 + 2\alpha_2(0) = \alpha_1$$

$$\therefore\ \alpha_1 = f'(a)$$

To get α_2 , we differentiate equation (3) twice and put $x = a$.

$$f''(x) = 2\alpha_2$$

$$\therefore\ \alpha_2 = f''\frac{(x)}{2}$$

Hence, from (3)

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

EXAMPLE 5 Find the quadratic approximation of the function $y = 2(1 + x)^{-1/2}$ about $a =$ 0.

SOLUTION

$$y = 2(1 + x)^{-1/2}$$

$$\mathrm{y} = \frac{2}{\sqrt{1 + x}}$$

$$f'(x) = \frac{0(\sqrt{1+x})) - 2.\frac{1}{2}(1+x)^{-1/2}}{(\sqrt{(1+x)})^2}$$

$$f'(x) = -\frac{1}{(1+x)^{\frac{3}{2}}} = -(1+x)^{-\frac{3}{2}}$$

$$f''(x) = \frac{3}{2}(1+x)^{-\frac{3}{2}-1} = \frac{3}{2}(1+x)^{-\frac{5}{2}}$$

Now, we know

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Here $a = 0$

$$f'(0) = -(1)^{-\frac{3}{2}} = -1$$

$$f''(0) = \frac{3}{2(1+x)^{\frac{5}{2}}} = \frac{3}{2}$$

$$f(0) = \frac{2}{\sqrt{1+0}} = 2$$

$$\therefore \frac{2}{\sqrt{1+x}} \approx 2 - 1(x-0) + \frac{3}{2(2)}x^2$$

$$= 2 - x + \frac{3}{4}x^2$$

9.5.3 Higher-Order Approximation of a Function

To further improve our approximation, we can use higher-degree polynomials.

For approximating a function $f(x)$ by a polynomial of degree n when x is close to a, we write

$$f(x) \approx \alpha_0 + \alpha_1(x-a) + \alpha_2(x-a)^2 + \cdots + \alpha_n(x-a)^n$$

Here,

$$\alpha_0 = f(a), \qquad \alpha_1 = \frac{f'(a)}{1!}, \qquad \alpha_2 = \frac{f''(a)}{2!}, \ldots, \alpha_n = \frac{f^n(a)}{n!}$$

$$\therefore f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

The RHS of this approximation is called the n^{th}-order Taylor polynomial for f about $x = a$.

EXAMPLE 6: Find the third-order Taylor Approximation for $f(x) = \frac{2}{x}$ about $a = 2$.

SOLUTION: The third–order approximation of a function is given by

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

Here,

$$f(x) = \frac{2}{x}$$

$$f'(x) = -\frac{2}{x^2} \quad \longrightarrow \quad f'(2) = -\frac{2}{4} = -\frac{1}{2}$$

$$f''(x) = \frac{0(x^2) - (-2)(2x)}{x^4} = \frac{4x}{x^4} = \frac{4}{x^3}$$

$$f''(0) = \frac{4}{8} = \frac{1}{2}$$

$$f'''(x) = \frac{0(x^3) - 4(3x^2)}{x^6} = -\frac{12}{x^4}$$

$$f'''(2) = -\frac{12}{2^4} = -\frac{12}{16} = -\frac{3}{4}$$

$$f(2) = \frac{2}{2} = 1$$

$$\therefore f(x) \approx 1 - \frac{1}{2}(x-2) + \frac{1}{2}(x-2)^2 - \frac{3}{4}(x-2)^3$$

IN-TEXT QUESTIONS

Choose the correct alternative:

2. The third–order Taylor approximation of $f(y) = (1+y)^{1/2}$ about $a = 0$ is:

   (a) $1 + \frac{1}{2}y - \frac{1}{4}y^2 + \frac{1}{8}y^3$

(b) $1+\frac{1}{2}y-\frac{1}{8}y^2+\frac{1}{16}y^3$

(c) $1-2y-4y^2-8y^3$

(d) $1-2y-8y^2-16y^3$

3. The radius of a spherical ball decreases from 10 to 9.8 cm. Then, the approximate decrease in its volume using linear approximation is:

(a) $20\pi\ cm^3$

(b) $40\pi\ cm^3$

(c) $60\pi\ cm^3$

(d) $80\pi\ cm^3$

4. If $\sqrt{x}+\sqrt{y}=1$, then $\left.\frac{dy}{dx}\right|_{x=\frac{1}{4},y=\frac{1}{4}}$ is:

(a) ¼

(b) 4

(c) ½

(d) 1

5. The quadratic approximation of $f(t)=(1+t)^5$ about $a=0$ is:

(a) $1+10t^2$

(b) $1+5t+20t^2$

(c) $1+5t+10t^2$

(d) $1-10t-5t^2$

9.6 ELASTICITIES

Economists often report elasticities instead of derivatives to report a change in a variable due to change in some other variable. This is because while derivatives are expressed in terms of units of the dependent and the independent variable, elasticities are unit–free.

For example, the slope of a demand curve tells us what the change in the quantity is demanded of a commodity due to a unit change in its price. For instance, what happens to the demand of ice-creams if its price per unit increases by \$1. While an increase of \$1 in the price of ice cream might cause a significant change in its quantity demanded, a price increase of \$1 may be inconsequential for a product such as an iPhone. Elasticities, on the other hand, are pure numbers devoid of any units. They tell us the percentage change in a dependent variable due to a unit change in the independent variable.

Suppose $y = f(x)$ is a differentiable function, then the elasticity of y with respect to x is given by:

$$El_x y = E_{yx} = El_x f(x) = f'(x)\frac{x}{y} = \frac{f'(x).x}{f(x)}$$

For instance, let $q = D(p)$ be the demand function for a commodity where q is the quantity demanded and p is the own price of the commodity. Then, the elasticity of demand of the commodity with respect to its price is

$$El_p q = \frac{d}{dp}\big(D(p)\big).\frac{p}{D(p)} = \frac{dq}{dp}.\frac{p}{q}$$

Note that for a linear demand curve, while the slope $\frac{dq}{dp}$ is the same at all the points along the curve, elasticity at all points is not the same.

EXAMPLE 7: Find the price elasticity of demand for the following function:

$$q = 200 - 4p\,, \quad at\; q = 40$$

SOLUTION:

$$q = 200 - 4p$$

$$\frac{dq}{dp} = -4$$

And,

$$El_q p = \frac{dq}{dp}.\frac{p}{q}$$

$$= (-4).\frac{p}{200 - 4p}$$

At $q = 40$,

$$40 = 200 - 4p$$
$$4p = 160$$
$$p = 40$$

$$\therefore El_q p = -4.\frac{40}{200 - 4(40)}$$
$$= -4.\frac{40}{40} = -4$$

IN-TEXT QUESTIONS

6. The elasticity of the demand function $q = \frac{8000}{p^{3/2}}$ at p=1 is:

(a) -2

(b) -3

(c) -0.5

(d) -1.5

9.7 SUMMARY

In this lesson, we learnt how to differentiate functions which cannot be expressed explicitly as a function of the independent variable. We then discussed the concept of differentials and their application in linear, quadratic, and higher-order approximations of functions. Finally, we discussed the concept of elasticities, followed by their applications in economics.

9.8 GLOSSARY

Differential: For a function $y = f(x)$, the term $dy = f'(x)dx$ is called the differential of the function. It is also denoted as df.

Elasticity: Elasticity of a function is the percentage change in a dependent variable due to a unit change in the independent variable.

Implicit functions: Functions in which the dependent variable cannot be expressed explicitly as a function of the independent variable are called Implicit functions.

9.9 ANSWERS TO IN-TEXT QUESTIONS

1. (b)
2. (b)
3. (d)
4. (d)
5. (c)
6. (d)

9.10 TERMINAL QUESTIONS

Q1. Given a demand function:

$$p = aq^b (a > 0)$$

i. Find the MR curve.
ii. Find the elasticity of demand.
iii. When will the elasticity be unity?
iv. What restriction on b should you impose?

Q2. The national income model of a closed economy is given by:

$$Y = C + I$$

$$C = f(y), \ \ where, 0 < f'(y) < 1$$

Here Y is national income, C is aggregate consumption and I is investment.

Under what condition will $\frac{d^2Y}{dI^2}$ be positive?

Q3. If $p = f(x)$, where $f'(x) < 0$, is an inverse demand function facing a monopolist, write total revenue (TR) as a function of output x. Find $\frac{d(TR)}{dp}$ by using chain rule and show that an increase in price leads to:

i. An increase in total revenue if the demand is inelastic.
ii. A decrease in total revenue if the demand is elastic.
iii. No change in total revenue if the demand is unitary elastic.

Q4. Find dy/dx for the following equations:

i. $xy = 100$
ii. $x^2 + xy - y^3 = 0$
iii. $x\sqrt{1+y} + y\sqrt{1+x} = 0$
iv. $y^3 - 6xy^2 = x^3 + 6x^2y$

Q5. Find the linear approximation of the following functions around the given points:

i. $y = x^2 + 4x + 3 \;\; around\ a = 2$
ii. $y = \frac{5x+2}{2x+7} \;\; around\ a = -1.$

Q6. If u and v are two differential functions of x, find:

i. $El_x(uv)$
ii. $El_x\left(\frac{u}{v}\right)$

9.11 REFERENCES

- Sydsaeter, K., Hammond, P. (2002). *Mathematics for economic analysis*. Pearson Education.
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LESSON 10

APPLICATIONS OF CONTINUITY AND DIFFERENTIABILITY

STRUCTURE

10.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to :

1. Develop an understanding for intermediate and mean value theorem.
2. Using Taylor's formula for polynomial approximation
3. Evaluating limit of intermediate forms using the L'Hôpital's Rule and
4. Compute inverse of a function.

10.2 INTRODUCTION

In this unit we will discuss certain theorems using the applications of continuity and differentiability. We will introduce intermediate – value theorem which forms the basis for optimization theory. Subsequently, we will also discuss the mean value theorem which is widely used in applications involving calculus. The later part of the unit presents Taylor's formulae used for approximation of polynomial and L'Hôpital's Rule for determining the limits of the intermediate forms. Finally, we give some details about the inverse function to conclude this unit.

10.3 THE INTERMEDIATE VALUE THEOREM

The intermediate value theorem is used to understand the concept of continuity. In economics, this theorem is applied to the concept of equilibrium. Now let us look at the theorem:

> "If we define a function $y = g(x)$ which is continuous for all x belong to the closed interval$[a, b]$, such that $a \leq x \leq b$ and $g(x)$ takes every value between $g(a)$ and $g(b)$ such that $g(a) \neq g(b)$." Sydsaeter & Hammond, 2009

This is known as the intermediate value theorem as any value that function $g(x)$ takes between $g(a)$ and $g(b)$ must occur for at least one number x between $x = a$ and $x = b$. More formally, it can be said that:

If $g(x)$ is a function continuous in $[a, b]$ and assuming that g(a) and g(b) have different signs, then $g(a) \neq g(b)$ then there is at least one real number c such that $a < c < b$ and $g(c) = 0$. (10.1)

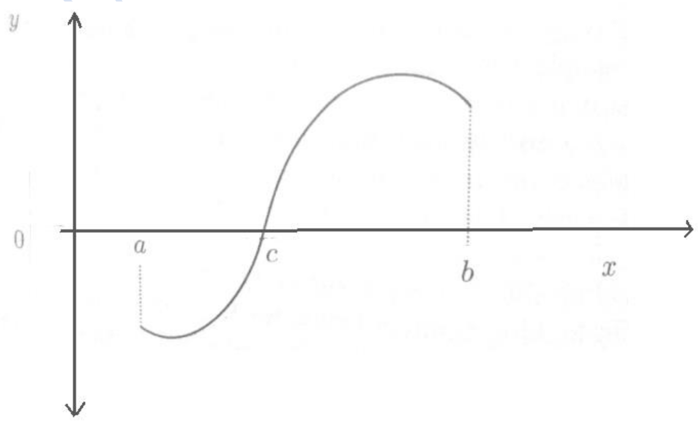


Figure 10.1: Illustrates the above theorem.

This theorem will not hold in the case of discontinuous functions. Fig. 10.1 illustrates the theorem. It is helpful in explaining the solutions to the equations where exact solutions cannot be obtained.

EXAMPLE: Prove that the equation $x^5 + 3x - 12 = 0$ has at least one solution between 1 *and* 2.

SOLUTION: The equation $g(x) = x^5 + 3x - 12 = 0$ is a polynomial. Further by theorem (10.1),

$$g(1) = (1)^5 + 3(1) - 12 = -8$$

While,

$$g(2) = (2)^5 + 3x - 12 = 35$$

Thus, there exists at least one number $c \in (1,2)$ such that $g(c) = 0$.

IN-TEXT QUESTIONS

1. Verify the following equations have at least one solution in the given interval:

   a) $x^6 + 3x^2 - 2x - 1 = 0 \ in \ (0,1)$

   b) $\sqrt{y^2 + 1} = 3y \ in \ (0,1)$

ANSWER TO IN-TEXT QUESTIONS

1. a) $f(0) = -1$ and $f(1) = 1$, there exists one solution.

   b) $f(0) = 0$ and $(1) = -1.6$, there exists at least one solution.

10.4 THE MEAN VALUE THEOREM

The mean value theorem explains the relationship between the derivatives of the functions and the slopes of the line. The statement of the theorem is:

If $g(x)$ is continuous in the closed and bounded interval $[a, b]$ and differentiable in (a, b)such that a *and* b are real numbers and $a < b$, then there exists a point c such that $a < c < b$ and

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

This theorem can be graphically illustrated in figure 10.2

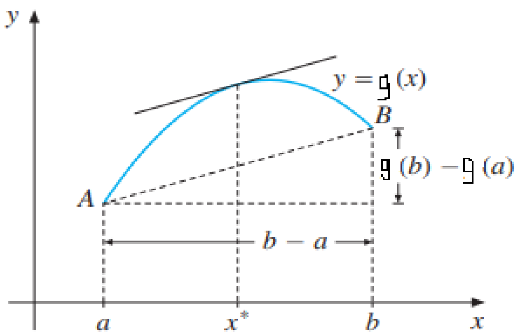


Figure 10.2

With the secant x_1x_2, there is a point C on the curve such that the tangent at this point is parallel to the curve. Let the coordinates of x_1, x_2 *and C* be represented as a, b, c respectively. The slope of this tangent line is $g'(c)$ such that

$$g'(c) = \frac{g(b)-g(a)}{b-a} \qquad (10.2)$$

thus, c satisfies the equation. In the case where, $g(a) = g(b) = 0$ *for all* $x \in (a, b)$, then there exists at least one root of $g'(x) = 0$. It is also known as *Rolle's Theorem*

EXAMPLE:_Verify the equation $f(x) = 5 + 6x - x^2$ satisfies mean value theorem in the interval [1,3].

SOLUTION: We find that

$$\frac{f(3) - f(1)}{3 - 1} = \frac{14 - 10}{2} = 2$$

And, on differentiating,

$$f'(x) = 6 - 2x$$

The equation

$$f'(x) = 6 - 2x = 2$$
$$\Rightarrow 4 = 2x$$
$$\Rightarrow x = 2$$

And 2 ϵ [1,3]. Thus, the mean value theorem is satisfied in this case.

Increasing & Decreasing Functions

A function g(x) is increasing in interval *I*, if $g(x_2) \geq g(x_1)$ whenever $x_2 > x_1$. Now using the concept of derivatives, we can say that if g(x) is increasing and differentiable, then $g'(x) \geq 0$. The mean value theorem can be used to make this precise and to prove the converse. Let *g* be a function which is continuous in the interval *I* and differentiable in the interior of *I* (that is, at points other than the end points). Suppose $g'(x) \geq 0$ for all x in the interior of *I*. Let $x_2 > x_1$ be any two numbers in *I*. According to the mean value theorem, there exists a number x* in (x_1, x_2) such that

$$g(x_2) - g(x_1) = g'(x^*)(x_2 - x_1)$$

$$g'(x^*) = \frac{g(x_2) - g(x_1)}{(x_2 - x_1)}$$

Because $x_2 > x_1$ and $g'(x*) \geq 0$, it follows that $g(x_2) \geq g(x_1)$, so g(x) is increasing. Thus, it can be followed that:

i. If $g'(x) > 0$ for all x in the interior of I, then $g(x)$ is strictly increasing in I.
ii. If $g'(x) \geq 0$ for all x in the interior of I, then $g(x)$ is increasing in I.
iii. If $g'(x) < 0$ for all x in the interior of I, then $g(x)$ is strictly decreasing in I.
iv. If $g'(x) \leq 0$ for all x in the interior of I, then $g(x)$ is decreasing in I.

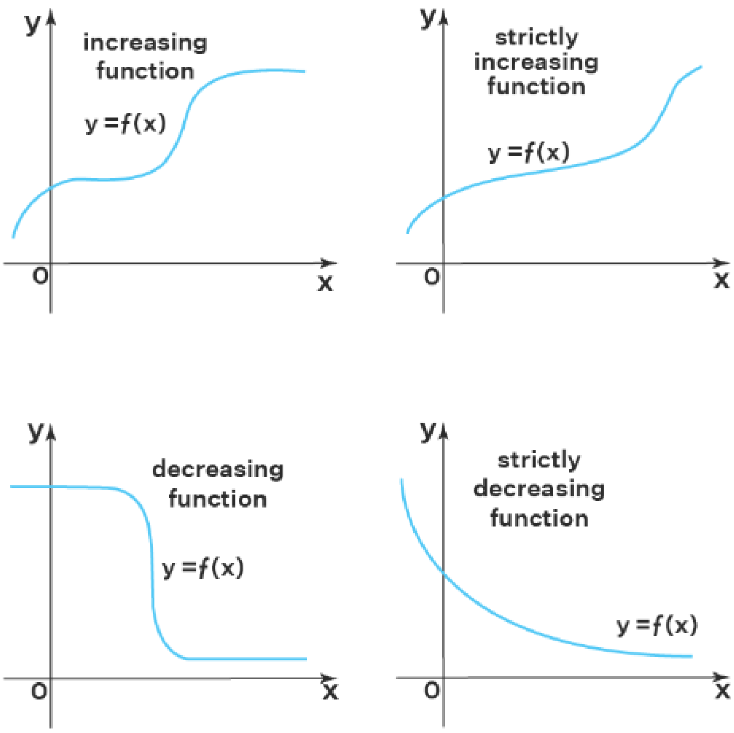


Figure 10.3: Behavior of a Function

In order to understand the increasing and decreasing conditions of the function, let us look at the graphs in Fig. 10.3. All the graphs above show increasing and decreasing function as we move from left to right along the graph.

IN-TEXT QUESTIONS

1. Verify that the following equations satisfy the mean value theorem:

   a) $f(x) = x^2 \ in \ [1,2]$

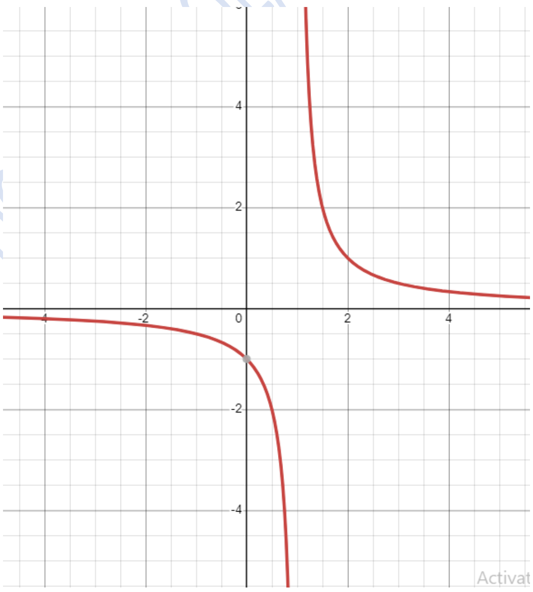
   b) $f(y) = \sqrt{a + y^2} \ in \ [0,4]$

2. Using the graph, show that mean- value theorem will hold for the function $y = \frac{1}{x-1}$ in the interval $[0,2]$.

ANSWER TO IN-TEXT QUESTIONS

1. a) $c = \frac{3}{2}$

   b) $c = \sqrt{3}$

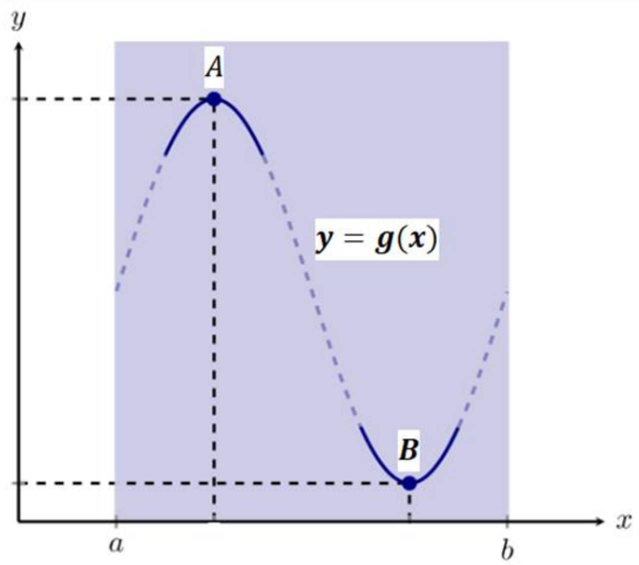


2. No, it will not hold.

10.5 THE EXTREME VALUE THEOREM

The extreme value theorem is used to determine the minimum and the maximum values of the function in an interval. The theorem states that if a function $g(x)$ is continuous in an interval (closed) $[a, b]$,then $g(x)$ has both minimum and maximum values on $[a, b]$.

While applying the theorem, we should keep in mind that the function should be defined on **closed and bounded interval.** The figure 10.4 depicts the theorem.



$figure$ 10.4

The graph is a continuous function with highest point A and lowest point B. The tangent to point A and B are parallel to $x - axis$. This implies that any of these two points, derivative of the function must be zero. In other words,

Let $g(x)$ be defined on the interval $[a. b]$and let $'c\,'$ be an interior point on this interval, then $'c\,'$ can be maximum or minimum point of $g(x)$ and if $g'(c)$ exists, then

$$g'(c) = 0 \qquad (10.3)$$

EXAMPLE: Does $f(x) = x^4 - 3x^3 - 1$ attains maximum or minimum value in $[-2,2]$?

SOLUTION: The function $f(x)$ is continuous on $[-2,2]$

And on differentiation,

$$f'(x) = 4x^3 - 9x^2$$

By the theorem (10.3),

$$f'(x) = 0$$

$$\Rightarrow 4x^3 - 9x^2 = 0$$

$$\Rightarrow x^2(4x - 9) = 0$$

$$\Rightarrow x = 0 \; or \; x = \frac{9}{4} = 2.25$$

Since, $x = 2.25$ is not in the interval $[-2,2]$, the only solution is $x = 0$.

The value of the function is

$f(0)$= -1

$$f(2) = (2)^4 - 3(2)^3 - 1 = -9$$

and $$f(-2) = (-2)^4 - 3(-2)^3 - 1 = 39$$

Therefore, it has minimum value as -9 at $x = 2$ and maximum value is 39 at $x = -2$.

10.6 APPROXIMATIONS

In the unit on differentiation, we briefly discussed the approximations of the polynomials. The concept of approximations is widely used in mathematics and economics. Economists built models to understand the theories and determine whether the results/ observations of these models closely approximate the real-world scenario.

In this section, we will introduce Taylor's Polynomial Expansion and Newton's Binomial Formulae.

10.6.1 Taylor's Approximation

If we consider a function $g(x)$ which we want to approximate over an interval at $x = a$ with a n^{th} degree polynomial, then it can be expressed as:

$$g(x) \approx g(a) + \frac{1}{1!}g'(a)x + \frac{1}{(2!)}g''(a)x^2 + ---- + \frac{1}{n!}g^{(n)}(a)x^n \qquad (10.4)$$

Using the same formulae, we write n^{th} $degree$ Taylor Polynomial for $g(x)$ near zero as i.e., $x = 0$ as $P(x)$

$$P(x) \approx g(0) + \frac{1}{1!}g'(0)x + \frac{1}{(2!)}g''(0)x^2 + ---- + \frac{1}{n!}g^{(n)}(0)x^n \qquad (10.5)$$

However, sometimes when making approximations, we need to ascertain how good our approximation is. In other words, we need to determine the error which is the difference between approximation and exact value. In equation 10.5, except at x = 0, function g(x) and the Taylor polynomial on the RHS of 10.5 will differ.

The difference between the two depends upon the value of x as well as $n,$ which is known as the remainder after n terms. It is denoted as $R_{n+1}(x)$. Thus, equation 10.5 becomes:

$$g(x) = g(0) + \frac{1}{1!}g'(0)x + \frac{1}{(2!)}g''(0)x^2 + ---- + \frac{1}{n!}g^{(n)}(0)x^n + R_{n+1}(x) \quad (10.6)$$

This leads us to a conclusion that if g is n + 1 times differentiable in an interval including 0 and x. Then the remainder $R_{n+1}(x)$ in equation 10.6 can be expressed as:

$$R_{n+1}(x) = \frac{1}{(n+1)!}\, g^{(n+1)}(c)(x)^{n+1} \quad (10.7)$$

For some number c between 0 $and\ x$.

This remainder formula is also known as $Lagrange's\ Form\ of\ Remainder$. It is the remainder form of the Taylor's expansion formula. Using 10.7 in equation 10.6, then

$$g(x) = g(0) + \frac{1}{1!}g'(0)x + \frac{1}{(2!)}g''(0)x^2 + ---- + \frac{1}{n!}g^{(n)}(0)x^n + \frac{1}{n+1!}g^{(n+1)}(c)x^{n+1} \quad (10.8)$$

10.6.2 Binomial Formulae

The binomial series also known as Newton's Binomial formula, is the Taylor's series with the functional form as $h(x) = (1+x)^k$ $where\ k\ is\ any\ real\ number.$

Using Taylor Formulae (10.4) in the case of binomial function for $x > -1$ is

$$h'(x) = k(1+x)^{k-1} \qquad h'(0) = k$$

$$h''(x) = k(k+1)(1+x)^{k-2} \qquad h''(0) = k(k-1)$$

Similarly,

$$h^n(x) = k(k-1) ----- [k-(n-1)](1+x)^{k-n}$$

$$h^n(0) = k(k-1) ----- [k-(n-1)]$$

Using these values and substituting in equation (10.4), we get

$$(1+x)^k = 1 + \frac{k}{1!}x + \frac{k(k-1)}{2!}\, x^2 -- + \frac{k(k-1) --[k-(n-1)]x^n\,]}{n!} + R_n(x)$$

(10.7)

Here,

$$R_{n+1}(x) = \frac{k(k-1)----(k-n)}{(n+1)!} x^{n+1} (1+p)^{k-n-1}$$

The R.H.S of the equation (1) can be expressed as generalized binomial coefficients

$$\binom{k}{r} = \frac{k(k-1)----(k-r+1)}{r!} \qquad (10.9)$$

Where, r is a Positive integer.

For Example,

$$\binom{\frac{1}{4}}{3} = \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{1.2.3} = \frac{7}{128}$$

Hence, the Newton's Binomial Formulae with k being an arbitrary real number and n is a positive integer is

$$(1+x)^k = 1 + \binom{k}{1} x + ---\binom{k}{n} x^n + \binom{k}{n+1} x^{n+1}(1+p)^{k-n-1} \quad (10.10)$$

Where, $0 < p < x \; and \; x > -1$.

EXAMPLE: Using binomial theorem, write down the expansion of $\sqrt[5]{33}$ and take n=2.

SOLUTION: Here, $x = 1/32, k = 1/5$ and taking $n = 2$, then

$$\sqrt[5]{33} = 2\left(1 + \frac{1}{32}\right)^{\frac{1}{5}}$$

$$\therefore \sqrt[5]{33} = 1 + 2\left(1 + \frac{1}{32}\right)^{\frac{1}{5}} = 2\left\{1 + \binom{\frac{1}{5}}{1}\left(\frac{1}{32}\right) + \binom{\frac{1}{5}}{2}\left(\frac{1}{38}\right)^2\right\}$$

$$= 2\left(1 + \frac{1}{5 \times 32} - \frac{2}{25} \times \left(\frac{1}{32}\right)^2\right) = 2.01$$

In the case where binomial function takes the general form, such as $h(x) = (\alpha + \beta)^K$ with k as a positive integer, then the Newton's binomial formula becomes:

$$(\alpha + \beta)^k = \alpha^k + \binom{k}{1}\alpha^{k-1}\beta + \binom{k}{2}\alpha^{k-2}\beta^2 + - - - - + \binom{k}{k}\beta^k. \quad (10.9)$$

EXAMPLE: Find the expansion of $(a + b)^6$.

SOLUTION: Using the above formula,

$$(a + b)^6 = a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

IN-TEXT QUESTION

1. Write down the binomial expansion of

   a) $(1 - 2x)^3$

   b) $(a + b)^4$

ANSWER TO IN-TEXT QUESTIONS

1. a) $1 - 6x + 12x^2 - 8x^3$

   b) $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

10.7 L'HÔPITAL'S RULE

Suppose we need to calculate the limit of a function where $x \to a$ and both numerator and denominator of the function tend to zero, such as

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \frac{0}{0}$$

Then, we call this limit as indeterminate form of type $0/0$. For dealing with these indeterminate forms, where $g(x)$ and $h(x)$ are both differentiable functions, we use the L hôpital's rule which comes with two versions. The **simpler version** states that:

If $g(x)$ and $h(x)$ are differentiable at c such that

$$g(c) = h(c) = 0$$

And,

$$h'(c) \neq 0$$

then,

$\lim_{x \to c} \frac{g(x)}{h(x)} = \frac{g'(x)}{h'(x)}$ (10.10)

This can be easily proved by mean value theorem that $g(x)$ and $h(x)$ are differentiable function then

$$\frac{g(x)}{x-c} \to g'(c) \qquad and \qquad \frac{h(x)}{x-c} \to h'(x)$$

As $x \to c$, and thus,

$\frac{g\prime(x)}{h\prime(x)} = \frac{g(x)/x-c}{h(x)/x-c}$ (10.11)

EXAMPLE: Using L Hôpital's Rule, determine

$$\lim_{y \to 0} \frac{(1+y)^{1/3} - 1}{y - y^2}$$

SOLUTION : Before solving the equation, we need to determine that whether it has an indeterminate form. Let numerator $g(y) = (1+y)^{1/3} - 1$and denominator is $h(y) = y - y^2$. Then,

$$g(0) = h(0) = 0$$

Next, we will compute $g'(y)$ and $h'(y)$ separately.

$g'(y) = \frac{1}{3}\ (1+y)^{-2/3}$ and $h'(y) = 1 - 2y$

Thus,

$$g'(0) = \frac{1}{3}\ and\ h'(0) = 1$$

And,

$$\frac{g(y)}{h(y)} = \frac{1}{3}$$

EXAMPLE: Find the following limit

$$\lim_{x\to 1}\frac{x-1}{x^2-1}$$

SOLUTION: Here, $g(x) = x - 1 \; and \; h(x) = x^2 - 1$

Also, $g(1) = h(1) = 0$

$$g'(x) = 1\,, h'(x) = 2x$$

Thus,

$$g'(1) = 1 \quad and \quad h'(1) = 2$$

So,

$$\frac{g(x)}{h(x)} = \frac{1}{2}$$

The **second version** of L Hôpital's rule, we will deal with a special case where $h'(c) = 0$. Suppose $g(c) = h(c) = 0 \; and \; h'(c) \neq 0$ if x is close but not equal to $'\,c\,'$ then,

$$\lim_{x\to c}\frac{g(x)}{h(x)} = \lim_{x\to c}\frac{g'(x)}{h'(x)}$$

given that the limit on the right-hand side exists. More formally,

If $g \; and \; h$ are differentiable in the interval (a, b) around c and $g(x) \to 0, h(x) \to 0 \; as \; x \to c$. If $h'(x) \neq 0$ for all $x \neq c$ in (a, b) then,

$$\lim_{x\to c}\frac{g(x)}{h(x)} = \lim_{x\to c}\frac{g'(x)}{h'(x)} = L \quad (10.12)$$

Where, L is a finite number.

EXAMPLE: Determine $L = \lim_{y\to\infty}(\sqrt[5]{y^5 - y^4} - y)$

SOLUTION: This can be written as:

$$\sqrt[5]{y^5 - y^4} - y = y\left(1 - \frac{1}{y}\right)^{1/5} - y = \frac{\left(1 - \frac{1}{y}\right)^{1/5} - 1}{\frac{1}{y}}$$

Thus, it becomes $0/0$ form as

$$\lim_{y\to\infty}\frac{\left(1-\frac{1}{y}\right)^{1/5}-1}{1/y}$$

Using L' Hôpital's Rule,

$$L=\lim_{y\to\infty}\left[-\frac{1}{5}\left(1-\frac{1}{y}\right)^{-\frac{4}{5}}\right]=\frac{1}{5}$$

In some cases, if h'(c) =0, then differentiate once more both denominator and numerator separately until the limit is determined.

IN-TEXT QUESTIONS

1. Find the following limits:

   a) $\lim_{x\to b}\frac{x^2-b^2}{x-b}$

   b) $\lim_{x\to 0}\frac{(1+4x)^{1/5}-1}{(1+5x)^{1/4}-1}$

2. Evaluate the following limit:

$$\lim_{y\to 2}\frac{y^4-4y^3+6y^2-8y+8}{y^3-3y^2+4}$$

ANSWER TO IN-TEXT QUESTIONS

1. a) $2b$

   b) 0.64

2. The second derivative gives:

$$\lim_{y\to 2}\frac{12y^2-24y+12}{6y-6}=\frac{12}{6}=2$$

10.8 INVERSE FUNCTION

An inverse function is a function that can reverse into another function. If say we consider a function as $p=f(x)$, then the inverse function will be x as a function of p such that $x=g(p)$.

For the existence of inverse functions, certain conditions have to be satisfied. The function $f(x)$ must be one-one. In other words, if $f(x)$ has a domain A and range B, then every element of the range B correspond to exactly one element in the domain A.The range element should not take two different values in the domain.

Graphically, it can be shown as Fig. 10.5,

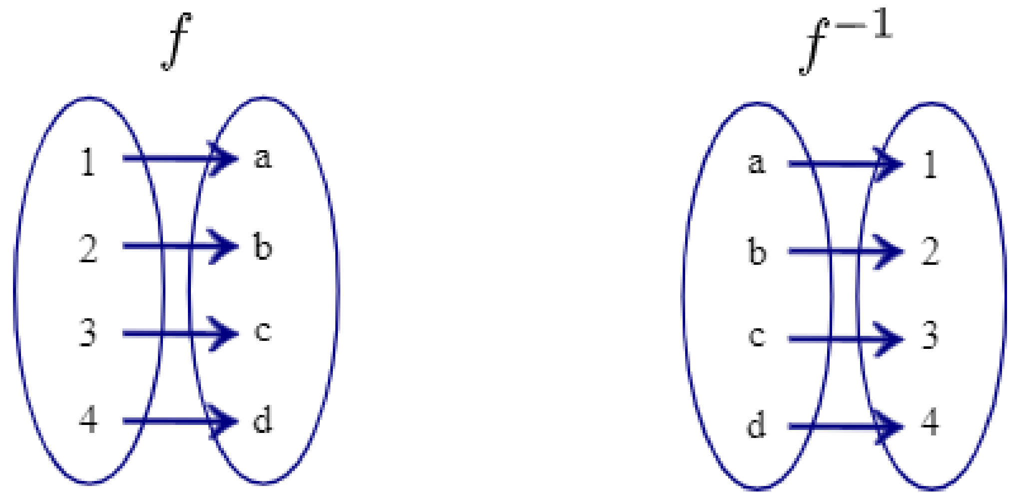


Figure 10.5: Inverse Functions

Here $f(x)$ maps from 1 to a while it's inverse $f^{-1}(x)$ maps a to 1. In general terms, the inverse of the function is defined as:

Let $g(x)$ be a function with domain A and range B. If $g(x)$ is one –to-one, then there exists its inverse function $h(x)$ with domain B and range A. For $g(x) = y$ and each $y \in B$, then value $h(y)$ has unique number $x \in A$.

$$h(y) = x \Leftrightarrow y = g(x) \quad \{x \in A, y \in B\}$$

While representing an inverse function, we often use the notation f^{-1}. However, one must not confuse f^{-1} with reciprocal of the other term $\frac{1}{f(x)}$.

EXAMPLE: Find the inverse function In terms of x:

a) $y = 2x + 3$

b) $y = \sqrt[3]{x + 1}$

SOLUTION: Solving these equations in terms of x,

a) $y = 2x + 3 \Leftrightarrow 2x = y - 3 \Leftrightarrow x = \frac{y-3}{2}$

b) $y = \sqrt[3]{x+1} \Leftrightarrow y = (x+1)^3 \Leftrightarrow y^3 = x+1 \Leftrightarrow x = y^3 - 1.$

GRAPHICAL DESCRIPTION OF INVERSE FUNCTIONS

When $g(x)$and $h(x)$ are inverse of each other, then their graphs $y = g(x) and\ h(x) = y$ are mirror images of each other with respect or symmetric to line $y = x$

Let us consider two functions:

$$g(x) = 2x - 3 \quad and \quad h(x) = \frac{1}{2}x + \frac{3}{2}$$

If we plot these graphs (Fig. 10.6) we will get figures like this:

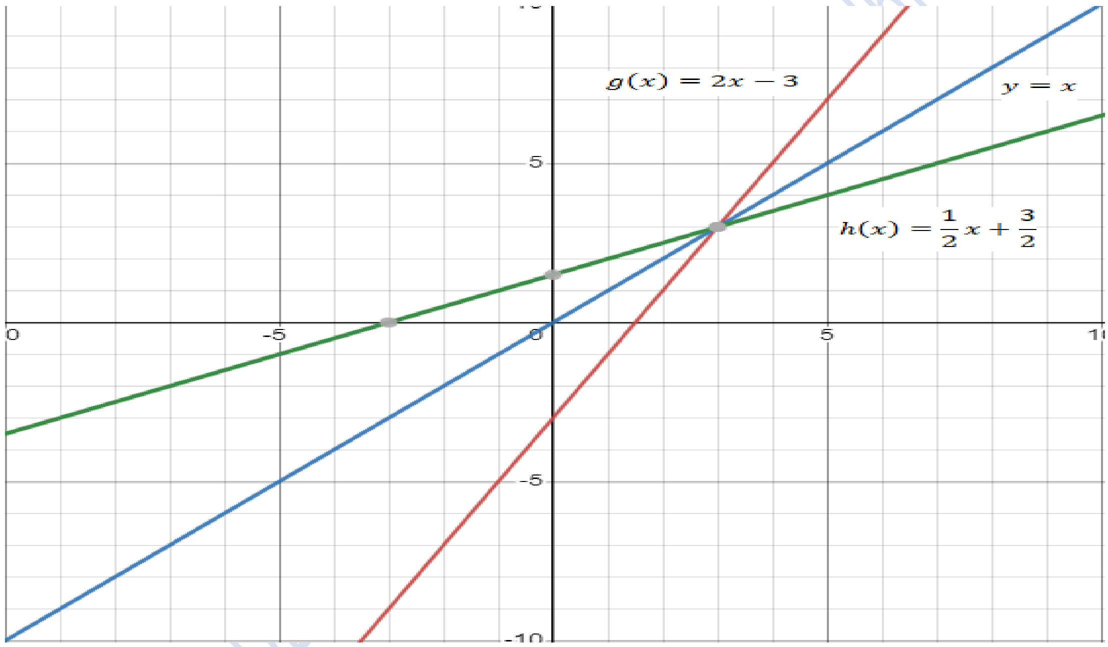


Figure 10.6

DERIVATIVE OF THE INVERSE FUNCTION

Assume that both the functions $g\ and\ h$ are differentiable, then $h\big(g(x)\big) = x$ can be differentiable with respect to x such that

$$h'\big(g(x)\big)g'(x) = 1 \qquad \{by\ chain\ rule\}$$

$$\Rightarrow h'g(x) = \frac{1}{g'(x)}$$

$$h'(y) = \frac{1}{g'(x)} \quad (10.13)$$

Thus, the inverse function rule states that the derivative of the inverse function is the reciprocal of the derivative of the original function. As observed from (10.13) that $g'(x)$ and $h'(x)$ has same sign. This indicates that either both functions are increasing (strictly) or both are decreasing (strictly). We can express the above conditions in a theorem.

THEOREM ON INVERSE FUNCTION

If g is continuous and strictly increasing (or strictly decreasing) function in an interval, A, then there exists an inverse function h that is continuous and strictly increasing (or strictly decreasing) in the interval(A) .

If x_1 is an interior point in the interval A and g is differentiable at x_1 with $g'(x_1) \neq 0$ then h is also differentiable at point $y_1 = g(x_1)$ and $h'(y_1) = \frac{1}{g'(x_1)}$

EXAMPLE: Find the inverse function h

$$g(x) = x^7 + 5x^5 + 2x - 2$$

SOLUTION: If $x_1 = 0 \ and \ y_1 = -2$,

We will get, $g(0) = -2$

And,

$$g'(x) = 7x^6 + 25x^4 + 2 > 0 \ for \ all \ x$$

Thus, $g(x)$ has an inverse h and $h'(-2) = \frac{1}{g'(0)}$

$$\Rightarrow h'(-2) = \frac{1}{2}$$

IN-TEXT QUESTIONS

1. The demand function faced by a monopolist is

$$q = \frac{1000 - p^3}{p^3}$$

Determine the inverse function of demand function.

2. Find the inverse of the following equations:

a) $g(y) = (y^3 - 1)^{1/3}$

b) $g(x) = \frac{x+1}{(x-2)}$

ANSWER TO IN-TEXT QUESTIONS

1. $p = \frac{10}{(q+1)^{1/3}}$

2. a) $h(y) = (y^3 + 1)^{1/3}$

   b) $h(x) = \frac{2x+1}{x-1}$

10.9 TERMINAL QUESTIONS

1. Let g be defined on the interval [0,1] and g (x)= $2x^2 - x^4$. Determine the range of g and find inverse function f of g.

2. Determine whether mean value theorem holds for the function y=$|x-3|$, $x \in [0,5]$.

3. Does function h satisfies the extreme value theorem with h is defined for all x ϵ (0,∞)

$$h(x) = \begin{cases} x+1 & x \in (0,1] \\ 1 & x \in (1,\infty) \end{cases}$$

4. For f(x) =$\frac{1}{1+x}$ use Taylor's expansion formula for n=2.

Answers

1. Range is [0,1] and inverse function f(x) = $\sqrt{1-\sqrt{1-x}}$
2. Mean value theorem does not hold.
3. Yes, it holds.
4. $1-x+x^2-(1+c)^{-4}x^3$

10.10 SUMMARY

This unit is the extension of the previous units on limits and continuity. We investigated how intermediate value theorem can be used to determine the solutions for those equations where exact solutions cannot be obtained. We then discussed the mean value theorem to explain the link between the derivative of the function and slope of the line. The unit also discussed approximations of the polynomial function using the results from Taylor's series and binomial series. Further, we looked how L Hôpital's rule can be used to determine the limits of a function with intermediate form. Finally, we conclude this unit with a brief description of the inverse functions.

10.11 REFERENCES

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LESSON 11

GLOBAL EXTREME POINTS

STRUCTURE

11.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to :

1. Explain the concept of global extreme points.
2. Apply the first derivative test for finding global extreme points.
3. Identify stationary points of functions and use them to find global extreme points and
4. Identify the applications of optimization techniques in economics.

11.2 INTRODUCTION

In almost all our daily activities, we try to find the best ways of achieving our desired outcomes. For instance, we use Google Maps to reach our destinations in the least possible time. An investor selects her portfolio of assets in such a way that she can generate maximum returns with minimum risk. Political parties decide their condition partners in such a way that they can maximize their chances of winning the election.

All these real-world situations are examples of what are called *optimization problems*. In economics, optimization has a key role to play since all the economic agents – households, firms and government are constantly engaged in trying to choose the best option among all the possible alternatives. For instance, firms must decide on the combination of inputs to maximize profits or minimize costs. Governments need to choose the optimal tax rate that would help maximize revenues.

To solve an optimization problem, we must first construct a mathematical model that best represents the problem statement. In most cases, such a mathematical model will be a function of several variables which will also be required to satisfy certain constraints.

In this lecture, however, we will restrict ourselves to unconstrained optimization problems that would involve maximization or minimization of functions of a single variable. The topics you will learn in this lesson will help you navigate through the more complex constrained optimization problems that will be covered in the subsequent course on mathematical methods for economics.

11.3 GLOBAL EXTREME POINTS

From the extreme value theorem studied earlier, we know that if a function is continuous in a closed and bounded interval[1], it attains both a maximum and a minimum value in the interval. The points at which the function attains these maximum and minimum values are called extreme points.

Formally, if a function $y = f(x)$ is defined over a domain D, then we say that

(a) $f(x)$ has a global maximum (maxima) at some point $c \in D$ if and only if

$$f(x) \le f(c) \quad \forall\ x \in D$$

$f(c)$ is then called as the maximum value of the function.

If $f(x) < f(c) \quad \forall\ x \in D$, then c is called as a strict maximum point.

(b) Similarly, $f(x)$ has a global minimum (minima) at some point $d \in D$ if and only if

[1] Recall that a closed interval is one that includes both of its endpoints. For a bounded interval, both the endpoints are real numbers.

$$f(x) \geq f(d) \quad \forall\ x \in D$$

If $f(x) > f(d) \quad \forall\ x \in D$, then d is called as a strict minimum point.

The points c and d are collectively called as optimal points or extreme points while the values of the function $f(x)$ at these points, i.e. $f(c)$ and $f(d)$ are known as optimal or extreme values.

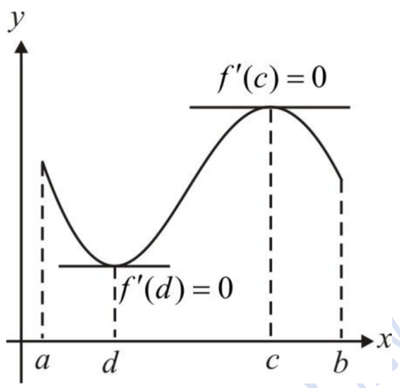


Figure 1

In Figure 1, the function $y = f(x)$ is defined over the domain [a, b]. The function attains a (Global) maxima at $x = c$ and a (global) minima at $x = d$.

11.4 CONCEPT OF STATIONARY POINTS

In figure 1, the function $f(x)$ attains the maximum and minimum values at $x = c$ and $x = d$, respectively. At both these points, the tangent to the graph of the function is parallel to the x-axis i.e. the slope of the functions is zero at both these points

$$f'(c) = f'(d) = 0$$

Points c and d are known as *stationary points*.

In general, a is a stationary point for a function f if the slope of the function at point a, is 0, i.e. $f'(a) = 0$.

Given the above discussion, we now state the following result without proof:

A necessary condition for $f(a)$ to be an extreme value of a function $f(x)$ is that $f'(a) = 0$ in case it exists.

Note that this result states that if the derivative exists, then it must be 0 at the extreme points. However, a function may attain an extreme value at a point without being differentiable at that point.

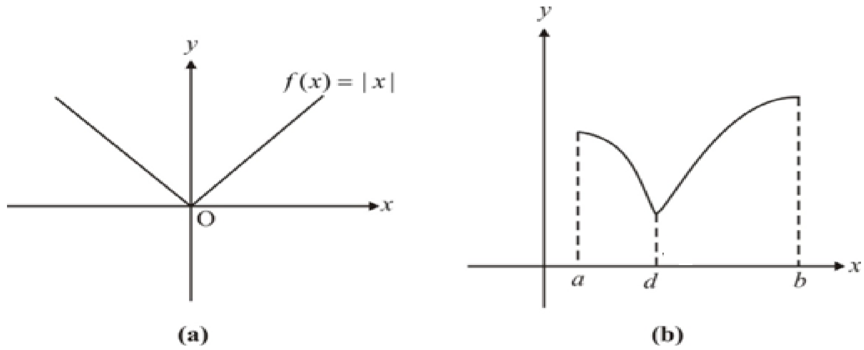


Figure 2

In figures 2(a) and 2(b), the functions attain minima at $x = 0$ and $x = d$ respectively without being differentiable at these points.

Further, the condition that the derivative of the function (if it exists) must be 0 at the extreme points is only a *necessary condition* for Maxima / Minima. It is not a sufficient condition. This basically means that even if $f'(x) = 0$ at $x = a$, a may not be an extreme point.

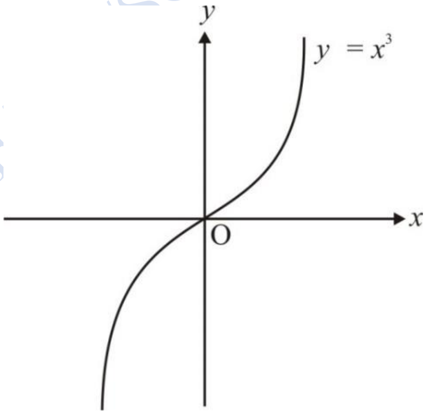


Figure 3

In figure 3, $f'(0) = 0$ but the function does not attain an extreme value at $x = 0$.

IN-TEXT QUESTIONS

1. Find the stationary points of the following functions:

(a) $f(x) = (x-5)^4$

(b) $f(x) = x^3 - 3x$

(c) $f(x) = x^3(x-1)^2$

(d) $f(x) = (x-1)(x+2)^2$

11.5 FIRST DERIVATIVE TEST FOR EXTREME POINTS

Suppose a function $f(x)$ is differentiable on an interval I and it has a stationary point at $x = c$. Then, by tracing the sign of $f'(x)$ to the left of c and the right of c, we can tell whether $x = c$ is a point of global maxima or minima.

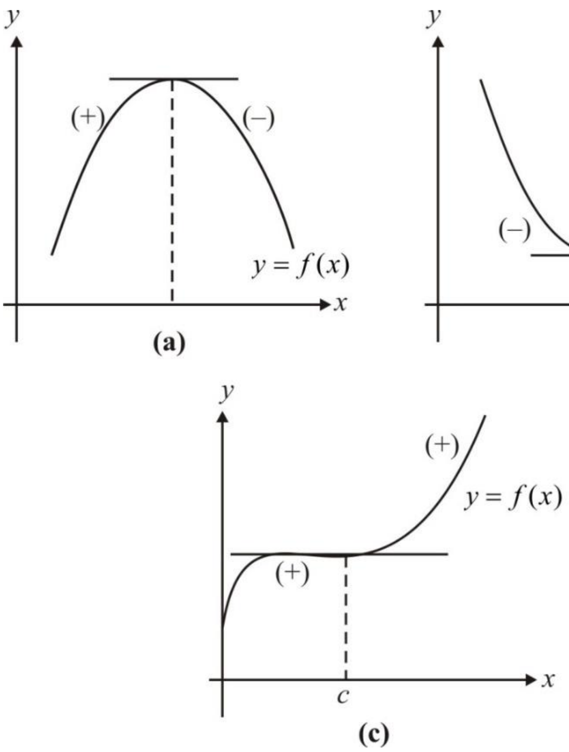


Figure 4

From Figure 4, we have the following results:

(i) In figure 4(a), the derivative of the function i.e., $f'(x)$ changes its sign from positive to negative at $x = c$ since $f(x)$ is increasing to the left of c and decreasing to the right of c. Here, $x = c$ is the maximum point for f.

Thus, if $f'(x) \geq 0$ for $x \leq c$ and $f'(x) \leq 0$ for $x \geq c$, then the function attains a global maxima at $x = c$.

(ii) In 4(b), the function is decreasing to the left of c and increasing to the right of c. In this case, the function attains a global minimum at $x = c$.

Thus, if $f'(x) \leq 0$ for $x \leq c$ and $f'(x) \geq 0$ for $x \geq c$, then the function attains a global minimum at $x = c$.

(iii) In figure 4(c), the derivative of the function does not change its sign from one side of c to the other because the function is increasing both to the left of c and right of c. Only the curvature of the function has changed at $x = c$. Here, c is known as an *Inflection Point*. As we will see in the subsequent lessons, inflection points are those points at which a function changes its curvature from convex to concave or vice-versa.

EXAMPLE 1 Find the extreme points (maxima/minima) and the extreme values of the following function:

$$g(y) = \frac{8y}{3y^2 + 4}$$

SOLUTION:

$$g(y) = \frac{8y}{3y^2 + 4}$$

$$g'(y) = \frac{8(3y^2+4) - 8y(6y)}{(3y^2+4)^2}$$

$$= \frac{24y^2 + 32 - 48y^2}{(3y^2 + 4)^2}$$

$$= \frac{32 - 24y^2}{(3y^2 + 4)^2} = \frac{8(4 - 3y^2)}{(3y^2 + 4)^2}$$

Now, the stationary points of the function are obtained by solving $g'(y) = 0$.

i.e., $4-3y^2=0$

$$y^2=\frac{4}{3}, \qquad y=\pm\frac{2}{\sqrt{3}}$$

Hence, the two stationary points are $y=\frac{-2}{\sqrt{3}}$ and $y=\frac{2}{\sqrt{3}}$. We now need to check what happens to the sign of $g'(y)$ to the left and right of these points.

Now, for $y\le\frac{-2}{\sqrt{3}}$, $g'(y)\le 0$ and for $y\ge-\frac{2}{\sqrt{3}}$, $g'(y)\ge 0$

Hence, $y=\frac{-2}{\sqrt{3}}$ is a point of minima and the minimum value of the function is given by:

$$g\left(\frac{-2}{\sqrt{3}}\right)=\frac{8\left(\frac{-2}{\sqrt{3}}\right)}{3\left(\frac{-2}{\sqrt{3}}\right)^2+4}=\frac{\frac{-16}{\sqrt{3}}}{3\left(\frac{4}{3}\right)+4}$$

$$=\frac{-16}{(\sqrt{3}/8)}=\frac{-2}{\sqrt{3}}=\frac{-2\sqrt{3}}{3}$$

Similarly, for $y\le\frac{2}{\sqrt{3}}$, $g'(y)\ge 0$ and for $y\ge\frac{2}{\sqrt{3}}$, $g'(y)\le 0$. Thus, $y=\frac{2}{\sqrt{3}}$ is a point of maxima and the maximum value of the function is given by

$$g\left(\frac{2}{\sqrt{3}}\right)=\frac{8\left(\frac{2}{\sqrt{3}}\right)}{3\left(\frac{2}{\sqrt{3}}\right)^2+4}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

IN-TEXT QUESTIONS

2. Given the following function:

$$f(x) = \frac{x}{x^2+4}, \; x \geq 0$$

Which of the following statements is true?

(a) The function attains a global minimum at x=2

(b) The function attains a global maximum at x= - 2

(c) x=2 is a point of inflection for the function

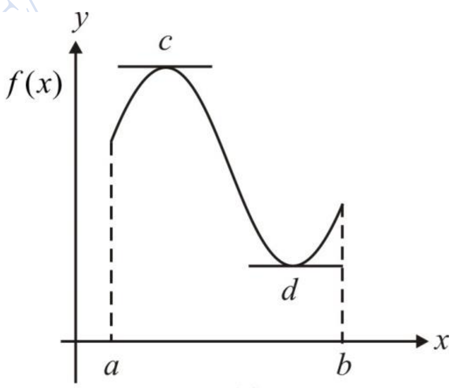
(d) The function attains a global maximum at x=2

11.6 ALTERNATIVE WAY OF FINDING EXTREME POINTS IN BOUNDED INTERVALS

In some cases, it may not be possible to locate points of maxima and minima by considering the sign variation of the first derivative. Hence, we must look at alternative methods to find the global extreme points in such situations.

Let us say we have a function $f(x)$ which is defined over a closed, bounded interval [a, b]. Then, the extreme points (Maximum/Minimum) of the function can be characterized by only one of the following 3 cases:

Case I: The extreme points are the Interior Points in $[a, b]$ where $f'(x) = 0$

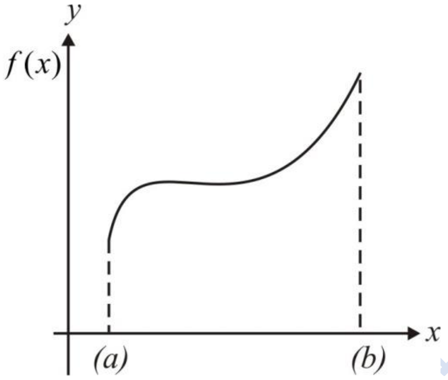


(c)

Here, points c and d are points of global maximum and global Minimum, respectively. Also, $f'(c) = 0$ and $f'(d) = 0$. Thus, if a global extremum (maximum/Minimum) occurs in the

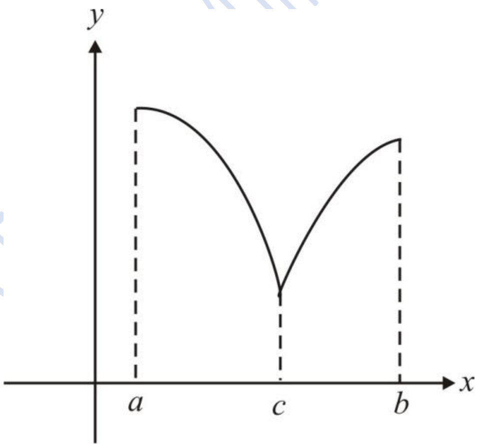
Interior of the interval on which the function is defined, the first derivative of the function is necessarily zero at that point.

Case-II: The extreme points are the end points of the Interval.



The function attains a global minimum at a and maxima at b but is not differentiable at these points.

Case-III: Maxima and Minima are in the interval of I $[a, b]$ but the function is not differentiable at these points.



The function attains a minimum at $x = c$ but is not differentiable at the point.

Given that most of the problems in economics involve functions that are usually differentiable everywhere, we would only consider problems of the type illustrated in case I and II.

To find the extreme points of a differentiable function f defined on a closed, bounded interval $[a, b]$, we follow the following steps:

Step I Find all stationary points in (a, b) i.e., $\forall\ x \in (a, b)$, find all points such that $f'(x) = 0$.

Step II Evaluate the functional value at all the stationary points.

Step III Find the value of the function at the end points i.e., at $x = a$ and $x = b$.

Step IV Compare the value of the function at the end points with the value of the function at the stationary points. The largest value is the global maxima, and the smallest value is the global minima.

Note that since we are assuming $f(x)$ to be differentiable everywhere, it is therefore continuous and hence by extreme value theorem we can say that the function does have global maximum and minimum points.

EXAMPLE 2. Find the global maximum and minimum values of the function $f(x) = 4x^2 - 40x + 80,\ x \in [0, 8]$.

SOLUTION $f(x) = 4x^2 - 40x + 80$

$$f'(x) = 8x - 40$$

For stationary $f'(x) = 0$.

$$8x - 40$$

$$x = 5$$

Now, $f(5) = 4(5)^2 - (40)(5) + 80$

$$= 100 - 200 + 80 = -20$$

Functional Value at End points

$$f(0) = 4(0) - 40(0) + 80 = 80$$

$$f(8) = 4(8^2) - 40(8) + 80$$

$$= 256 - 320 + 80 = 16$$

Thus, the function attains a (global) maxima at $x = 0$ and the maximum value of the function is 80.

The function attains a (global) minima at $x = 5$ and the minimum value of the function is –20.

11.7 ECONOMIC APPLICATION OF OPTIMIZATION: PROFIT MAXIMIZATION

We know that the profit function of a firm is given by

$$\pi = TR - TC$$

where TR is the total revenue and TC is the total cost associated with x units of output. Thus, TR and TC are both functions of x.

Hence, the total profit also becomes a function of x. We can, therefore, write,

$$\pi(x) = TR(x) - TC(x)$$

Now, because of capacity constraints, there is a maximum quantity $\bar{x}$ that can be produced by the firm in a given period.

Hence, $x \in [0, \bar{x}]$

Assuming that TR(x) and TC (x) are differentiable functions of x, the profit function π is also differentiable and continuous in $[0, \bar{x}]$ and hence has a maximum value.

In some cases, while the maximum profit might occur at $x = 0$ or $x = \bar{x}$, we will assume for now that the maximum value of the function occurs at $x = x^*$ where x^* is an interior point in $[0, \bar{x}]$.

Thus, at x^*, $\pi'(x^*) = 0$

Since, $\pi(x) = TR(x) - TC(x)$

$$\pi'(x^*) = TR'(x^*) - TC'(x^*)$$

For Maxima

$$\pi'(x^*) = 0$$

$$TR'(x^*) = TC'(x^*) \Rightarrow MR(x^*) = MC(x^*)$$

Now, TR' is nothing but the Marginal Revenue i.e., addition to the total revenue when an extra unit of output is sold while TC' is the cost of producing an additional unit of output i.e., the Marginal Cost.

Hence, for profit maximization, the production should be undertaken till a point where Marginal revenue is equal to the Marginal cost.

Perfect Competition Case: In the case of perfect competition, the firm has no control over the price. Hence, $TR(x) = P \cdot x$ in the case of Perfect competition, where P denotes the price per unit of the output x.

$$\therefore \ MR(x) = \frac{dTR(x)}{dx} = P$$

So, for Profit Maximization,

$$MR(x) = MC(x)$$

$$P = MC(x)$$

Thus, in the case of perfect competition, the profit maximizing output is at the point where price per unit of the output is equal to the Marginal cost.

EXAMPLE 3. The total cost of producing x units of a commodity is given by $C(x)$. Assuming that $C(x)$ is differentiable, prove that $AC(x)$ has a stationary point at some $x = x^*$ if and only if the marginal cost and the average cost are equal at x^*.

SOLUTION

We can define average cost (AC) as

$$AC(x) = \frac{C(x)}{x}$$

Taking x^* to be an interior point, we need to prove that

$$AC'(x^*) = 0 \Leftrightarrow AC(x^*) = MC(x^*)$$

Since this an if and only if statement, we need to prove two propositions:

(I) $AC'(x^*) = 0 \Rightarrow AC(x^*) = MC(x^*)$

(II) $AC(x^*) = MC(x^*) \Rightarrow AC'(x^*) = 0$

I Part

$$AC(x) = \frac{TC(x)}{x}$$

$$AC'(x) = \frac{TC'(x)x - TC(x)\cdot 1}{x^2}$$

$$= \frac{TC'(x)\cdot x}{x^2} - \frac{TC(x)}{x^2}$$

$$= \frac{MC(x)}{x} - \frac{AC(x)}{x}$$

At $x = x^*$

$$AC'(x^*) = 0$$

$$\frac{MC(x^*)}{x^*} - \frac{AC(x^*)}{x^*} = 0$$

$$\Rightarrow \quad \frac{MC(x^*)}{x^*} = \frac{AC(x^*)}{x^*}$$

$$MC(x^*) = AC(x^*)$$

II Part

Given $AC(x^*) = MC(x^*)$ (1)

Dividing the equation throughout by x^*

$$\frac{AC(x^*)}{x^*} = \frac{MC(x^*)}{x^*} \quad (2)$$

Since $AC(x) = \frac{TC(x)}{x}$

$$AC'(x) = \frac{TC'(x)\cdot x - TC(x)\cdot 1}{x^2} = \frac{MC(x)}{x} - \frac{AC(x)}{x}$$

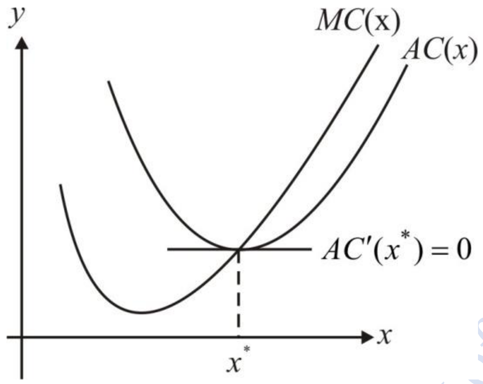
At $x = x^*$

$$AC'(x^*) = \frac{MC(x^*)}{x^*} - \frac{AC(x^*)}{x^*}$$

$= 0$ (Using (2))

Hence proved.

We can also illustrate the result graphically:



Thus, the average cost function attains a minimum at the point where *AC* = *MC*.

IN-TEXT QUESTIONS

3. Assume that due to capacity constraints the maximum amount that can be produced of a commodity x is 500. Suppose the market is perfectly competitive with the price per unit of the commodity equal to 1840. The cost function of the commodity is $\pi(x) = 2x^2 + 40x + 5000$. Then, the profit maximizing value of the output is:

   (a) 0

   (b) 500

   (c) 450

   (d) 1,000

11.8 SUMMARY

In this lesson, we discussed the meaning of global extreme points of a function. We learnt how we can find these extreme points by studying the sign variations of the function's first derivative. For functions which are differentiable everywhere, we learnt that the extreme values can also be found by comparing the values of the function at the end points of the interval over which it is defined and the stationary points. Finally, we discussed some applications of optimization (maximization and minimization) in economic theories.

11.9 GLOSSARY

Global Maximum: If a function $f(x)$ is defined over a domain *D*, then $f(x)$ has a global maximum at some point $c \in D$ iff $f(x) \leq f(c) \quad \forall\ x \in D$.

Global Minimum: If a function $f(x)$ is defined over a domain *D*, then $f(x)$ has a global minimum at some point $c \in D$ iff $f(x) \geq f(d) \ \forall\ x \in D$

Stationary Points: The values of x at which $f'(x)$ = 0 are called as stationary values or stationary points.

11.10 ANSWERS TO IN-TEXT QUESTIONS

1. (a) 5 (b) 1, -1 (c) 0, 1, 3/5 (d) 0, -2
2. (d) The function attains a global maximum at x=2.
3. (c) 450

11.11 TERMINAL QUESTIONS

Q1) Find the global extreme (Minimum / Maximum) points and the maximum/minimum values of the following functions by using the first derivative test:

(a) $f(t) = -2.05 + 1.06t - 0.04t^2, \ t \geq 0$

(b) $f(y) = 5(y+2)^4 - 3$

(c) $g(h) = -13.62 + 0.984h - 0.05h^{3/2}, \quad h \geq 0$

(d) $f(y) = \dfrac{-2}{2+y^2}$

Q2) Find the maximum and minimum values of the function over the indicated interval:

(a) $f(y) = 2y^3 - 24y + 107, \qquad y \in [1,3]$

(b) $f(t) = \left(\dfrac{1}{2} - t\right)^2 + t^3, \quad t \in [-2, 2.5]$

(c) $f(w) = (w-2)\sqrt{w-1}, \ w \in [1,9]$

(d) $f(s) = \dfrac{s^2+1}{s}, \quad s \in \left[\dfrac{1}{2}, 2\right]$

Q3) A person has income of m_1 and m_2 in periods 1 and 2, respectively. His utility function is given by

$$\ln a_1 + \frac{1}{1+\theta} \ln a_2$$

where a_1 and a_2 are his consumption levels in periods 1 and 2, respectively while θ is the discount rate. The person either saves or borrows in period 1 at a rate of interest r. Find the value of a_1 that maximizes his utility function.

Q4) The total cost function of a commodity is $C(q) = 1{,}000 + 100q - 10q^2 + q^3$. Show that the average cost function in the case has a stationary point q_0 where the marginal cost and average cost are equal.

11.12 REFERENCES

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- Hoy, M., Livernois, J., McKenna C., Rees, R., Stengos, T. (2001). *Mathematics for Economics*, Prentice-Hall India.

LESSON 12

LOCAL EXTREME POINTS

STRUCTURE

12.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to :

1. Explain the concept of local extreme points.
2. Apply the first derivative test to find local extreme points.
3. Apply the second derivative test to identify local extreme points and
4. Illustrate the use of local optimization techniques in economic applications.

12.2 INTRODUCTION

In the previous lesson, we discussed methods to find out the points of global maximum and minimum of a function i.e., the points at which the function attains its largest and smallest values, respectively. For finding the global extreme points, we compared the function values at all points in the domain. However, in some optimization problems, we are also interested in finding out the points at which the function attains a maximum or minimum in their immediate neighborhood. These are called local or relative extreme points.

12.3 LOCAL EXTREME POINTS

Consider the following figure:

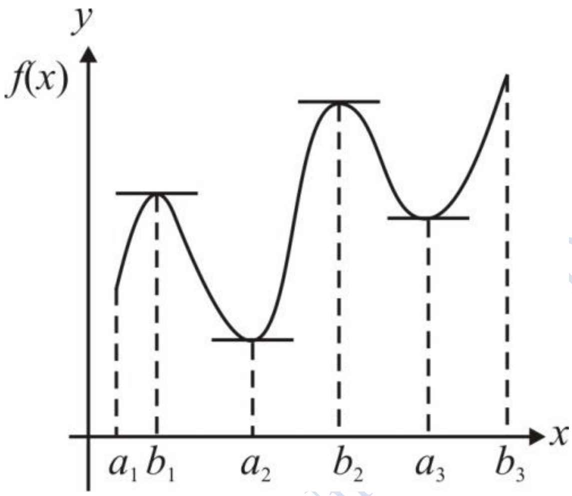


Figure 1

In figure 1, the values of the function at points b_1, b_2 and b_3 are greater than the values of the function in the neighborhood of these points. Points b_1, b_2, b_3 are therefore referred to as points of local maximum. Similarly, the values of the function at points a_1, a_2 and a_3 are less than the values of the function in the neighborhood of these points. Points a_1, a_2, a_3 are therefore referred to as points of local minimum.

Formally, we define local maximum and local minimum as follows:

(1) A function $f(x)$ is said to attain a local maximum at x = a if there is an interval (a_1, a_2) around a such that $f(x) \leq f(a)$ for all those x in the domain D of $f(x)$ that also lie in (a_1, a_2).

(2) A function $f(x)$ is said to attain a local minimum at x= b if there is an interval (b_1, b_2) around b such that $f(x) \geq f(b)$ for all those x in the domain D of $f(x)$ that also lie in (b_1, b_2).

Note that these definitions are also applicable for the endpoints of the function. Hence, in Figure 1, the endpoint a_1 is a local minimum point while the endpoint b_3 is a (global) local maximum point.

The values of the function at the local extreme points are called local extreme values.

12.4 FIRST DERIVATIVE TEST FOR FINDING LOCAL EXTREME POINTS

Like in the case of global extreme points, the local extreme points can either be the interior points of the interval over which the function is defined, the endpoints of the interval or the points in the interval where the function is not differentiable.

We consider the case where the local extreme points are in the interior of the interval I and the function is differentiable (and hence continuous) throughout. Hence, from the previous lesson, we know that the derivative of the function at the local extreme points must be zero. Recall that the points at which the derivative of a function is zero are called stationary points.

Hence, in order to find the local extreme values of a function, the first step is to find the stationary points. We then need to decide whether these points are points of local maximum, local minimum or neither. In order to do that, we study the sign variations of the first derivative of the function around the intervals of the stationary points. The detailed steps are given below:

Consider a function $y = f(x)$ and suppose a is a stationary point for the function. Then,

(i) If $f'(x) \geq 0$ throughout some interval (a_1, a) to the left of a and $f'(x) \leq 0$ throughout some interval (a, a_2) to the right of a, then $x = a$ is a point of local maxima for $f(x)$

(ii) If $f'(x) \leq 0$ throughout some interval (a_1, a) to the left of a and $f'(x) \geq 0$ throughout some interval (a, a_2) to the right of a, then $x = a$ is a point of local minima for $f(x)$.

(iii) If $f'(x) > 0$ both throughout some interval (a_1, a) to the left of a and throughout some interval (a, a_2) to the right of a, then $x = a$ is not a local extreme point for $f(x)$. This is because if $f'(x) > 0$ in (a_1, a) and in (a, a_2), then $f(x)$ is strictly increasing throughout the interval around a. The same result holds if $f'(x) < 0$ i.e., the function is strictly decreasing on both sides of $x = a$.

EXAMPLE 1 Use the first derivative test of find the points of local maximum/minimum (if any) of the following functions:

(a) $f(x) = x^3 - 12x$

(b) $f(y) = (y-1)^3(y+1)^2$

SOLUTION:

(a) $f(x) = x^3 - 12x$

$$f'(x) = 3x^2 - 12$$

Put $f'(x) = 0$

$$3x^2 - 12 = 0 \Rightarrow 3(x^2 - 4) = 0$$

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2.$$

Thus, $x = 2$ and $x = -2$ are the two stationary points of the function.

We now analyse the changes in signs of $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$ to the left and right of the stationary points as shown below:

	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$x > 2$
Sign of $f'(x)$	+	0	–	0	+

Since $f'(x) \geq 0$ for $x \leq -2$ and $f'(x) \leq 0$ for $x \geq -2$, the function attains a local maximum at $x = -2$. Further, $f'(x) \leq 0$ for $x \leq 2$ and $f'(x) \geq 0$ for $x \geq 2$. Therefore, the function attains a local minimum at $x = 2$

(b)

$f(y) = (y-1)^3(y+1)^2$

$$f'(y) = 3(y-1)^2(y+1)^2 + (y-1)^3 \times 2(y+1)$$

$$= (y-1)^2(y+1)[3(y+1)+2(y-1)]$$

$$= (y-1)^2(y+1)[3y+3+2y-2]$$

$$= (y-1)^2(y+1)(5y+1)$$

For local maxima/ minima, $f'(y) = 0$

$\Rightarrow \quad (y-1)^2(y+1)(5y+1)$

$y = 1$ or $y = -1$ or $y = -1/5$

We now analyse the sign variations in $f'(x)$ around these stationary points as shown below:

	$y < -1$	$y = -1$	$-1 < y < -1/5$	$y = -1/5$	$-1/5 < y < 1$	$y = 1$	$y > 1$
Sign of $f'(x)$	+	0	–	0	+	0	+

As is evident from the sign diagram, $f'(y)$ changes its sign from Positive to Negative as it passes through $y = -1$. Hence, $y = -1$ is a point of local maximum for $f(y)$.

Similarly, $f'(y)$ changes its sign from negative to positive as it passes through $y = -1/5$. Hence, $y = -1/5$ is a point of local minima for $f(y)$. However, $f'(y)$ does not change its sign as it passes through $y = 1$. Hence, $y = 1$ is neither a point of local maximum nor a point of local minimum.

IN-TEXT QUESTIONS

Choose the correct alternative:

1. Let $f(t) = t^3 - 6t^2 + 12t - 8$. Then, $f(t)$ has

   (a) a local maximum, at $t = 2$.

   (b) a local minimum at $t = 2$

   (c) Neither a maximum nor a minimum, at $t = 2$

   (d) None of these

2. Let $f(y)=(y-2)^4(y+1)^3$. Then, the function has:

   (a) a local maximum at $y = 2/7$

   (b) a local minimum at $y = 2$.

   (c) a point of inflection at $y = -1$

   (d) all of the above.

3. The local minimum value of the function $(x-5)^4$ is

   (a) 5

   (b) 0

   (c) –5

   (d) 2

4. The average cost function $AC = q^2 - 5q + 8$ level of:

   (a) 5

   (b) 0

   (c) 2.5

   (d) None of the above

12.5 THE SECOND DERIVATIVE TEST FOR LOCAL EXTREME POINTS

The use of the first derivative test to determine whether a stationary point is a local maximum, a local minimum or neither requires the knowledge about the sign variations of the first derivative. An alternative method to characterize the nature of the stationary points is using the second derivative test which only requires knowledge about the properties of the function at the stationary points.

The *Second-Derivative Test* is stated as follows:

Let $f(x)$ be a twice differentiable function in an interval I. Suppose a is an interior point of I and $f'(a) = 0$ (i.e., a is a stationary point). Then,

(i) $f''(a) < 0 \Rightarrow a$ is a strict local maximum point.

(ii) $f''(a) > 0 \Rightarrow a$ is a strict local minimum point.

(iii) $f''(a) = 0 \Rightarrow a$ can either be a point of local maximum or minimum or inflection.

Proof of (i)

Given that $f'(a) = 0$ and $f''(a) < 0$. Since $f''(a)$ is the derivative of $f'(x)$ at $x = a$, we can write

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}$$

$$= \lim_{h \to 0} \frac{f'(a+h)}{h} \qquad [\because f'(a) = 0]$$

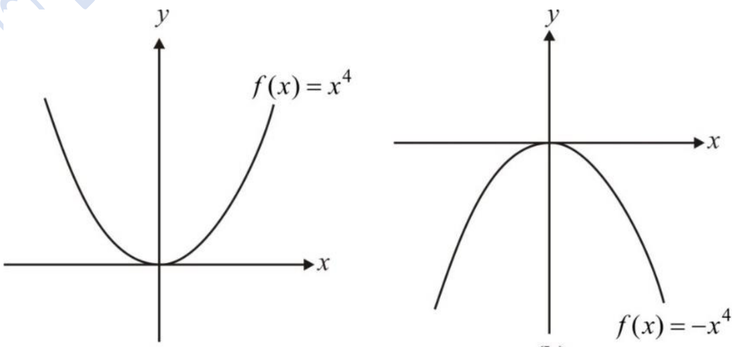
$$f''(a) < 0 \Rightarrow \lim_{h \to 0} \frac{f'(a+h)}{h} < 0 \qquad (1)$$

Now, if h is a small positive number, then equation (1) implies that $f'(a+h) < 0$. Then, $f'(x)$ is negative in an interval to the right of a. Similarly, if h is a small negative number, then from equation (1), $f'(a+h) > 0$. Thus, $f'(x)$ is positive in an interval to the left of a. Hence, by first derivative test, a is a point of strict local maximum.

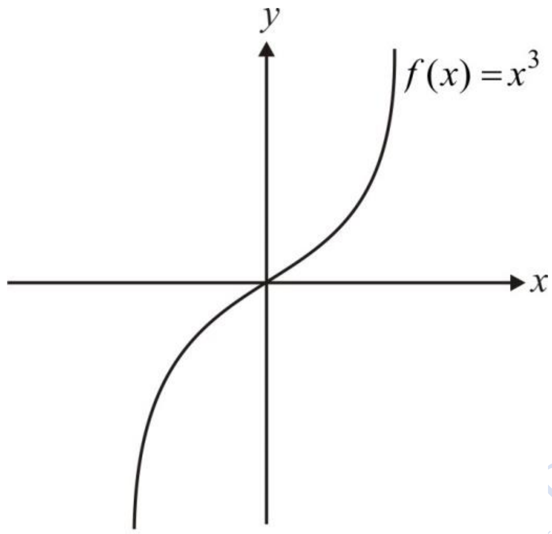
We can prove part (ii) of the second derivative test similarly.

Part (iii) is discussed below:

Consider the functions $f(x) = x^4$, $f(x) = -x^4$, $f(x) = x^5$. Each of these functions satisfy $f'(x) = 0$ and $f''(x) = 0$ at $x = 0$. However, the first function attains a local minimum at $x = 0$, the second function attains a local maximum at $x = 0$ while for the third function $x = 0$ is a point of inflection.



(a) (b)



(c)

Figure 2

Hence, if $f'(x) = 0$ and $f''(x) = 0$, the second-order derivative test is inconclusive and hence we must either apply the first-derivative test or resort to some other test in such cases.

Necessary Versus Sufficient Conditions

The condition of $f'(x)$ being equal to 0 is a necessary condition in both the first derivative test and the second-derivative test. Since, it is based on the first-order derivative of $f(x)$ it is also referred to as the *first-order necessary condition*. However, it is worth reiterating that it is a necessary condition but not a sufficient condition. This means that even if $f'(x) = 0$ at some point $x = a$, it does not mean that the function $f(x)$ attains a local maximum or minimum at a. For example, if $f(x) = x^3$, $f'(x) = 0$ at $x = 0$, but it is neither a point of local maximum or local minimum for the function as illustrated in figure 2(c). In fact, $x = 0$ is a point of inflection for $f(x) = x^3$.

Now, once the first-order necessary condition is satisfied at $x = a$, i.e. $f'(x) = 0$ at $x = a$, the negative (positive) sign of the second-order derivative $f''(x)$ is sufficient to establish that the point a is a point of local maximum (minimum). The conditions $f''(x) < 0$ and $f''(x) > 0$ are referred to as second-order sufficient conditions. Note that these conditions are sufficient but not necessary. This means that a function $f(x)$ can attain a local maximum/minimum at $x = a$ without the second order derivative being positive or negative at this point. For example, if $f(x)=x^4$, then $f'(x)=0$ at x = 0 and $f''(x)=0$ at x = 0. However, the function attains a local minimum at x = 0.

We can therefore restate the conditions for local maxima and minima as follows:

Condition	Local Maxima	Local Minima
First-order necessary	$f'(x)=0$	$f'(x)=0$
Second order necessary	$f''(x) \leq 0$	$f''(x) \geq 0$
Second-order sufficient	$f''(x) < 0$	$f''(x) > 0$

The second-order necessary conditions consider the fact that a local maximum/minimum can occur not only when $f''(x)$ is negative / positive but also when $f''(x)$ is zero.

EXAMPLE 2 Use the second-derivative test to find the points of local maximum/ minimum (if any) for the following functions:

$$f(y)=2y^3-21y^2+36y-20$$

SOLUTION

$$f(y)=2y^3-21y^2+36y-20$$

$$f'(y)=6y^2-42y+36$$

$$f'(y)=0$$

$$\Rightarrow 6y^2-42y+36=0$$

$$6(y^2-7y+6)=0$$

$$6(y^2-6y-y+6)=0$$

$$6(y(y-6)-1(y-6))=0$$

$$6(y-1)(y-6)=0$$

y = 1, 6 are the stationary points.

We shall now evaluate the second-order derivatives at these points:

$$f''(y) = 12y - 42$$

$$f''(1) = 12 - 42 = -30 < 0$$

$\therefore$ y = 1 is a point of local maximum.

$$f''(6) = 12(6) - 42 = 72 - 42 = 30 > 0$$

$\therefore$ y = 6 is a point of local minimum

IN-TEXT QUESTIONS

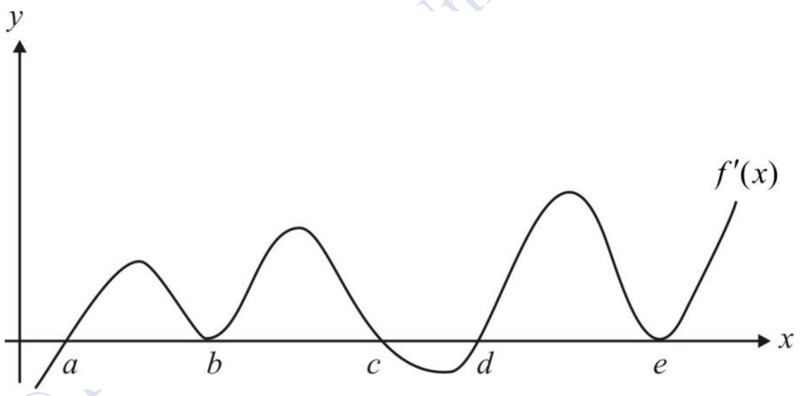
Choose the correct alternative:

5. If the function $f(x)$ is known to attain a local minimum at $x = a$, then we must have $f''(a) > 0$. This statement is :

   (a) True

   (b) False

6. Suppose the graph of the first-order derivative of a function is given by



   Then, which of the following is true:

   (a) The function attains local maximum at *a* and *d*.

   (b) The function attains local minimum at *a* and *d*.

   (c) *c* is a local maximum point for *f*

   (d) Both (ii) and (iii)

12.6 ECONOMIC APPLICATIONS

In the previous lesson, we discussed the problem of profit maximization in economics. The total profit was defined as:

$$\pi(x) = TR(x) - TC(x)$$

where x denotes output.

The first-order condition for optimization was derived as:

$$\pi'(x) = TR'(x) - TC'(x) = 0$$

$$TR'(x) = TC'(x)$$

$\Rightarrow$ MR = MC

Now, in certain cases, it might happen that solving for MR= MC may give several solutions. Hence, in such cases, we need to characterize the nature of the stationary points. We can do so by using the second-derivative test.

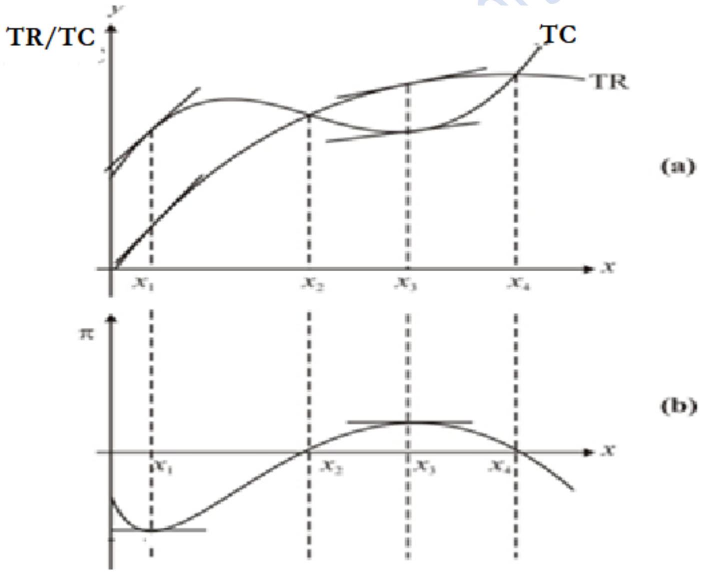


Figure 3

Figure 3(a) shows the total revenue (TR) and total cost (TC) curves. Figure 3(b) shows the profit curve which is the difference between the TR & TC curves. From figure(b), we can see that the first-order necessary condition for profit maximization is satisfies at two output levels x_1 and x_3 since at these output levels $\frac{d\pi}{dx} = 0$ because the tangent to the curve is parallel to the

x-axis at the points. Further, as we can see from figure 3(a), at both the output levels x_1 and x_3, $MR = MC$ because the tangent lines drawn to the two curves of TR & TC are parallel to each other at x_1 and x_3. However, at an output level of x_1, the producer is actually incurring a loss since $TC > TR$ at x_1. The profit is actually getting maximized at the output level of x_3.

At x_3, the second-order derivative $\left.\frac{d^2\pi}{dx^2}\right|_{x=x_3}$ is negative. At x_1, the second order derivative is positive. Hence, by the second-order sufficient conditions, we can say that the profit is getting maximized only at the output level of x_3.

EXAMPLE 3 Find the profit maximizing level of output if the total revenue (TR) and total cost (TC) functions are:

$$TR(x) = 1200x - x^2$$

$$TC(x) = x^3 - 61.25x^2 + 1{,}582.5x + 2{,}000$$

SOLUTION:

$$\pi(x) = TR(x) - TC(x)$$

$$= 1200x - x^2 - x^3 + 61.25x^2 - 1582.5x - 2{,}000$$

$$= -x^3 + 59.25x^2 - 328.5x - 2{,}000$$

$$\pi'(x) = \frac{d\pi}{dx} = -3x^2 + 118.5x - 328.5 = 0$$

Hence, $x^* = 3,\ 36.5$

$$\frac{d^2\pi}{dx^2} = -6x + 118.5$$

$$\left.\frac{d^2\pi}{dx^2}\right|_{x^*=3} = -6(3) + 118.5 = 100.5 > 0$$

$$\left.\frac{d^2\pi}{dx^2}\right|_{x^*=36.5} = -6(36.5) + 118.5 = -100.5 < 0$$

Thus, from the second-order derivative tests, the profit maximizing level of output is 36.5 units.

12.7 SUMMARY

In this lesson, we discussed the concept of local extreme points. We learnt how to identify the local extreme points by studying the sign variations of the stationary points in their respective neighborhoods. We also discussed the method of finding the local extreme points by using the second derivative test. Finally, we discussed the use of local optimization techniques in the profit maximization exercise of a firm. \

12.8 GLOSSARY

Local Maximum: A function $f(x)$ attains a local maximum at $x = a$ if there is an interval (a_1, a_2) around a such that $f(x) \leq f(a)$ for all those x in the domain D of $f(x)$ that also lie in (a_1, a_2).

Local Minimum: A function $f(x)$ attains a local maximum at $x = b$ if there is an interval (b_1, b_2) around b such that $f(x) \geq f(b)$ for all these x in the domain D of $f(x)$ that also lie in (b_1, b_2).

Local Extreme Values: The values of the function at the local extreme points are called as local extreme values.

12.9 ANSWERS TO IN-TEXT QUESTIONS

1. (c)
2. (d)
3. (b)
4. (c)
5. (b)
3. (d)

12.10 TERMINAL QUESTIONS

Q1) Determine the local extreme points and the corresponding local extreme values for the following functions:

(i) $f(x) = x^3 - 3x + 8$

(ii) $f(y) = y^3 - 6y^2 + 12y - 8$

(iii) $f(t) = (t-5)^4$

(iv) $f(s) = s^3(s-1)^2$

Q2) What requirements must be imposed on the constants α, β and γ in the following function.

$$f(x) = x^3 + \alpha x^2 + \beta x + \gamma$$

so that the function has a local minimum at $x = 0$?

Q3) Find the profit-maximizing level of output for a firm whose demand function is given by $q = 100 - p$ and cost function is given by

$$C = \frac{1}{3}q^3 - 7q^2 + 111q + 50$$

Q4) Let the utility function of an individual defined for a day i.e. 24 hours be:

$$U = 48L + Ly - L^2$$

where L denotes Leisure, y denotes income from work and w denotes wage rate. Show that the number of hours of work for maximum utility are given by

$$w^* = \frac{12w}{1+w}$$

Further, show that irrespective of how high the wage rate becomes, the individual will never be prepared to work for more than 12 hours a day.

12.11 REFERENCES

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LESSON 13

CONVEX AND CONCAVE FUNCTIONS

STRUCTURE

13.1 LEARNING OBJECTIVES

After reading this lesson, students will be able to:

1. Understand the interpretation of the second derivative of a function.
2. Distinguish between convex and concave functions and their characteristics.
3. Analyse the signs of the second-order derivatives to classify a function as concave / convex.
4. Explain the meaning and types of inflection points and
5. Elucidate the role of concavity and convexity of functions in economic models.

13.2 INTRODUCTION

In the last lesson, we saw how the second derivative of functions could be analyzed to find the local extreme points. In this lesson, we will discuss the interpretation of the second derivative and its application in describing the curvature of functions and classifying them as convex and concave functions. The distinction between convexity and concavity of a function is also extremely essential in many economic models.

13.3 INTERPRETATION OF THE SECOND-ORDER DERIVATIVE OF A FUNCTION

The first derivative of a function $f'(x)$ measures the rate of change of the function $f(x)$. Further, the sign of the first derivative determines whether a function is increasing or decreasing.

$$f'(x) \geq 0 \text{ on } (a_1, a_2) \Leftrightarrow f(x) \text{ is increasing on } (a_1, a_2) \tag{1}$$

$$f'(x) \leq 0 \text{ on } (a_1, a_2) \Leftrightarrow f(x) \text{ is decreasing on } (a_1, a_2) \tag{2}$$

By the same logic, the second derivative $f''(x)$ measures the rate of change of the first derivative $f'(x)$. Hence,

$$f''(x) \geq 0 \text{ on } (a_1, a_2) \Leftrightarrow f'(x) \text{ is increasing on } (a_1, a_2) \tag{3}$$

$$f''(x) \leq 0 \text{ on } (a_1, a_2) \Leftrightarrow f'(x) \text{ is decreasing on } (a_1, a_2) \tag{4}$$

In other words, $f''(x)$ measures the rate of change of the rate of change of the original function $f(x)$.

Hence, given a function $f(x)$, the first derivative of the function at a point $x = a$ tells us about the sign of the slope of the function at that point (positive/negative). Meanwhile the second derivative of the function at point $x = a$ tells us whether the slope is increasing or decreasing at that point.

For instance, if at a point $a \in D$, $f'(a) > 0$ and $f''(a) > 0$, it means that the slope of the curve at the point $x = a$ is positive and increasing. This means that the value of the function is increasing at an increasing rate at x=a. Similarly, if $f'(a) > 0 \; but \; f''(a) < 0$, it means that the value of the function is increasing at a decreasing rate.

While similar results hold when the first derivative is negative, we must be careful while interpreting the change in the values of the negative slope. For example, if $f'(a) <$

0 *but* $f''(a) > 0$, it means that the slope of the curve at the point x=a is negative and increasing. For example, this means that the slope is changing from – 6 (a smaller number) to –5 (a larger number). Graphically, this means that the negative slope becomes less steep as x increases. Similarly, $f'(a) < 0$ and $f''(a) < 0$ implies that the slope of the curve at the point $x = a$ is negative and decreasing. As an example, consider a case where the slope of a function is changing from –15 (a larger number) to – 16 (a smaller number). Here, the negative slope becomes steeper as x increases.

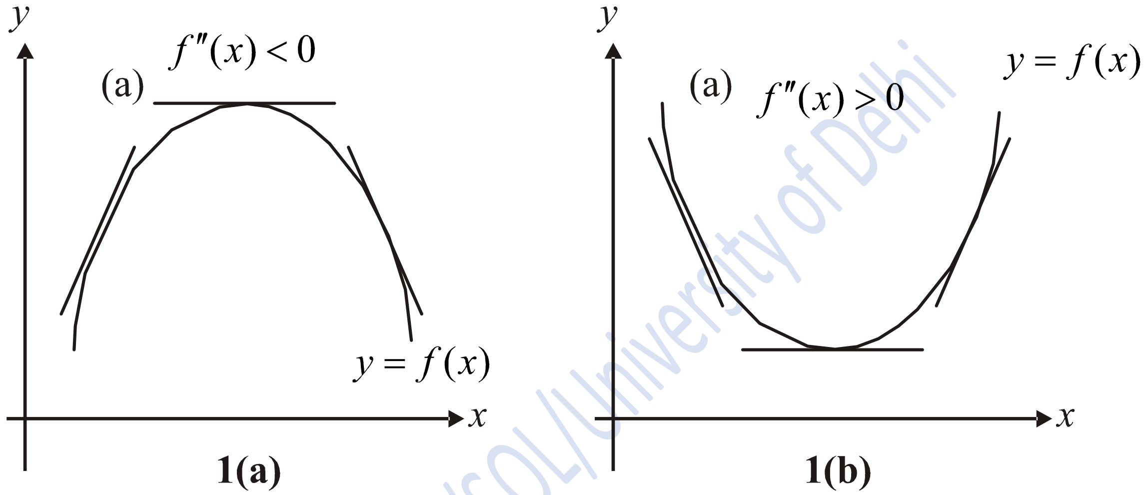


Figure 1

In figure 1(a), the slope of the curve decreases with an increase in the value of x. It is a situation where $f''(x)$ is decreasing throughout. Meanwhile in figure 1(b), the slope of the curve increases as x increases. This is a situation where $f''(x) > 0$ throughout.

From Figure 1, we can see that the second-order derivative of a function $f''(x)$ helps us characterize the curvature of its graph. The graph in Figure 1(a) is that of a concave function while that in 1(b) of that convex function.

Formally we say that if a function $f(x)$ is continuous in the interval I and twice differentiable in the interior of I (denoted by I°), then:

$f(x)$ is convex on I $\Leftrightarrow$ $f''(x) \geq 0 \quad \forall\, x \in I^{\circ}$

$f(x)$ is concave on I $\Leftrightarrow$ $f''(x) \leq 0 \quad \forall\, x \in I^{\circ}$

Based on the signs of the first order and second-order derivatives, the following four cases are possible:

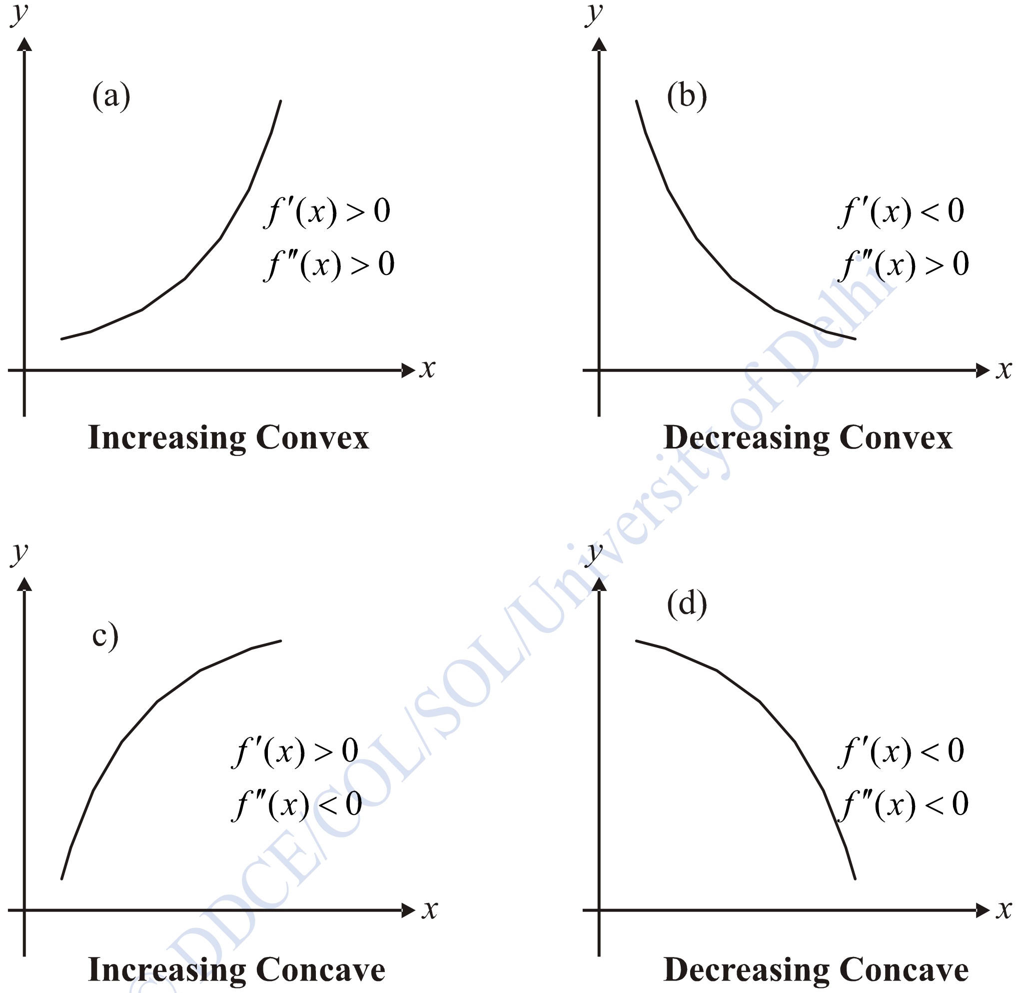


Figure 2

Note that for a linear function such as $f(x) = 2x + 3$, the second-order derivative $f''(x)$ is equal to 0. Hence, a linear function is both concave as well as convex. We, therefore, have the following definitions of strict convexity and concavity.

$f(x)$ is strictly convex on I $\Leftrightarrow$ $f''(x) > 0 \ \forall x$ in the interior of I.

$f(x)$ is strictly concave on I $\Leftrightarrow$ $f''(x) < 0 \ \forall x$ in the interior of I.

It should be clear from the definitions of convexity and strict convexity that a strictly convex function is automatically a convex function, but the converse is not true. A similar result holds in the case of concave and strictly concave functions.

EXAMPLE 1 Check the concavity / convexity of

$$f(x) = -\frac{1}{3}x^2 + 8x - 3$$

SOLUTION: $f(x) = -\frac{1}{3}x^2 + 8x - 3$

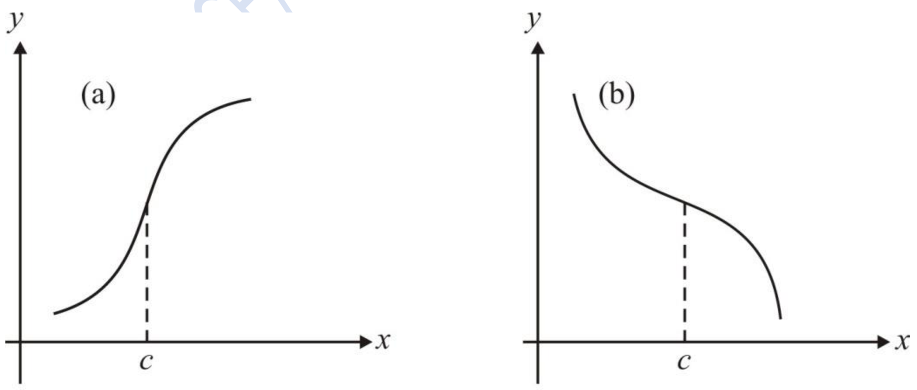
$$f'(x) = -\frac{2}{3}x + 8$$

$$f''(x) = -\frac{2}{3} < 0 \quad \forall x$$

Hence, $f(x)$ is concave (strictly concave).

13.4 INFLECTION POINTS

In many economic applications, the functions that we study are often not convex or concave throughout. They may be convex in some parts of the domain but concave in others (in vice-versa). The points at which a function changes its curvature from convex to concave (or vice-versa) are called *Inflection Points*. Figure 3 illustrates the nature of inflection points in different cases.



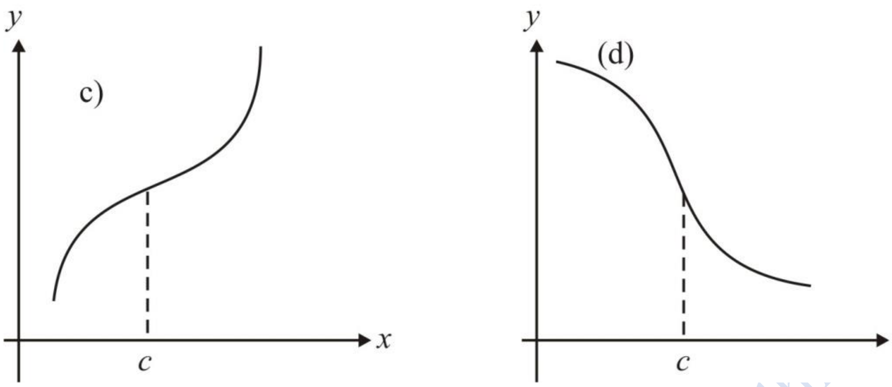


Figure 3

In figure 3(a) – (d), the graph of the function changes its curvature at the point c. In figures 3(a) and 3(b), the graph changes from being convex to concave at the point c while in figures 3(c) and 3(d), the graph changes from being concave to convex at the point c. Thus, c is a point of inflection in all the four cases.

Formally, we define an inflection point as follows:

Suppose $y = f(x)$ is a twice-differential function. Then, point c is an inflection point if there is an interval (a, b) containing c such that either of the following two conditions hold:

(a) $f''(x) \geq 0$ if $a < x < c$ (i.e., to the left of c) and

$f''(x) \leq 0$ if $c < x < b$ (i.e., to the right of c).

(b) $f''(x) \leq 0$ if $a < x < c$ (i.e., to the left of c) and

$f''(x) \geq 0$ if $c < x < b$ (i.e., to the right of c).

In case (a), the second derivative i.e., $f''(x)$ changes its sign at c from positive to negative and the curve changes from being convex to concave. Here, c is called the point of inflection of the first-class.

In case (b), the second derivative $f''(x)$ changes its sign at c from negative to positive and the curve changes from being concave to convex. Here, c is called the point of inflection of the second class.

13.4.1 Test for Inflection Points

Let f be a function with a continuous second derivative in an interval I. Suppose that c is an interior point of I. Then,

(a) If c is an inflection point for f, then $f''(c) = 0$.

(b) If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f.

While part (b) follows from the definition of an inflection point, part (a) can be proved as follows:

Since c is an inflection point for *f*,

$\therefore$ $f''(x) \leq 0$ on one side of c and $f''(x) \geq 0$ on the other,

$\therefore$ $f''(c) = 0$.

Note that $f''(c) = 0$ is a necessary condition for c to be an inflection point but not a sufficient condition. This means that $f''(c) = 0$ does not imply that $f''(x)$ changes sign at $x = c$.

For Example: Consider the function $y = x^4$.

Here, $f'(x) = 4x^3$

$f''(x) = 12x^2$

$f''(x) = 0$ at x = 0.

However, 0 is not an inflection point for $f(x)$ because $f''(x) > 0 \ \forall \ x \neq 0$ and hence does not change its sign at $x = 0$.

EXAMPLE 2. Find the point of inflection of (if any) of the function

$$f(x) = x^3 - 15x^2 + 20x + 20$$

SOLUTION: $f(x) = x^3 - 15x^2 + 20x + 20$

$$f'(x) = 3x^2 - 30x + 20$$

$$f''(x) = 6x - 30 = 6(x - 5)$$

A necessary condition for the existence of an inflection point is $f''(x) = 0$.

$\Rightarrow \quad 6(x-5)=0 \Rightarrow x=5$

Now we can see that $f''(x) > 0$ for $x > 5$ and $f''(x) < 0$ for $x < 5$.

Thus, at $x = 5$, $f''(x) = 0$ and $f''(x)$ also changes its sign. Hence, $x = 5$ is a point of inflection for the given function.

IN-TEXT QUESTIONS

Choose the correct alternative:

1. The function $f(x)=\frac{1}{a}x^3-\frac{1}{6}x^2-\frac{2}{3}x+1$ has an inflection point at

   (a) $x=0$

   (b) $x=1/2$

   (c) $x=1$

   (d) $x=2$

2. For what values of α and β, does the following function pass through (–1,1) and reaches an inflection point at $x = ½$,

   $f(x)=\alpha x^3+\beta x^2$

   (a) $\alpha=-2/5, \quad \beta=3/5$

   (b) $\alpha=2/5, \quad \beta=3/5$

   (c) $\alpha=-2/5, \quad \beta=-3/5$

   (d) $\alpha=2/5, \quad \beta=-3/5$

3. The function $f(t)=\frac{1-t}{1+t}$ has

   (a) an inflection points at x = –1

   (b) an inflection points at x = 1

   (c) an inflection points at x = 0

   (d) no inflection points.

13.5 MAXIMUM / MINIMUM FOR CONCAVE / CONVEX FUNCTIONS

Suppose $f(x)$ is a concave function is an interval I and c is a stationary point for $f(x)$ in the interior of I, then c is a maximum point for f(x) in I.

Reason: Since $f(x)$ is a concave function in the interval I, $f''(x) \leq 0 \ \forall \ x \in I$. This implies that $f'(x)$ is decreasing in the interval I. Now, if c is a stationary point in I, then $f'(c) = 0$.

Since $f'(x)$ is decreasing and $f'(c)$ = 0, this is only possible if $f'(x) \geq 0$ to the left of c and $f'(x) \leq 0$ to the right of c. Hence, $x = c$ is a maximum point for $f(x)$ in I am using the first-derivative test.

By the same logic, we can say that if $f(x)$ is a convex function in an interval I and c is a stationary point in the interval of I, then c is a minimum point for $f(x)$ in I.

EXAMPLE 3 Find the intervals in which the following cubic cost function is convex and where it is convex. Also, find the unique inflection point.

$$C(Q) = aQ^3 + bQ^2 + cQ + d$$

where $a > 0, b < 0, c > 0, d > 0, b^2 < 3ac$.

Also, sketch the graph of the function

SOLUTION

Given $C(Q) = aQ^3 + bQ^2 + cQ + d$

$$C'(Q) = 3aQ^2 + 2bQ + c$$

$$C''(Q) = 6aQ + 2b$$

Test for Inflection Point

$$C''(Q) = 0$$

$$6aQ + 2b = 0 \Rightarrow Q = \frac{-b}{3a}$$

Now, we can see that, for

$$Q \in \left(0, \frac{-b}{3a}\right), C''(Q) < 0$$

Hence, $f(x)$ is strictly concave in the interval $\left(0, \frac{-b}{3a}\right)$.

For $\quad Q \in \left(\frac{-b}{3a}, \infty\right), C''(Q) > 0$

Hence, $f(x)$ is strictly convex in the interval $\left(\frac{-b}{3a}, \infty\right)$.

Thus, $Q = \frac{-b}{3a}$ is an inflection point.

Graph of the function

(a) If $Q = 0,\ C(Q) = d$. This is the total fixed cost, which is the cost that is incurred even when the output produced is zero.

(b) $C'(Q) = 3aQ^2 + 2bQ + c$

$$= 3a\left(Q^2 + \frac{2bQ}{3a} + \frac{c}{3a}\right)$$

$$= 3a\left(Q^2 + \left(\frac{1}{2} \times \frac{2b}{3a}\right)^2 - \left(\frac{1}{2} \times \frac{2b}{3a}\right)^2 + \frac{2bQ}{3a} + \frac{c}{3a}\right)$$

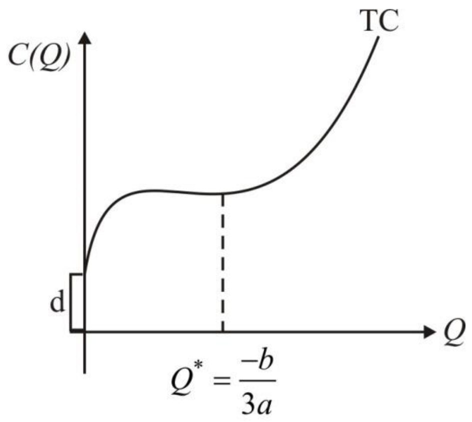
$$= 3a\left(Q^2 + \left(\frac{b}{3a}\right)^2 - \left(\frac{b}{3a}\right)^2 + \frac{2bQ}{3a} + \frac{c}{3a}\right)$$

$$= 3a\left(\left(Q + \frac{b}{3a}\right)^2 - \frac{b^2}{9a^2} + \frac{c}{3a}\right)$$

$$= 3a\left(\left(Q + \frac{b}{3a}\right)^2 - \left[\frac{b^2 + 3ac}{9a^2}\right]\right)$$

Given that a >0 and $b^2 < 3ac$, $f'(x) > 0$. Therefore, the function is strictly increasing throughout its domain.

Hence, using all the above information, the graph of the function is :



IN-TEXT QUESTIONS

Choose the correct alternative:

4. If $f(x) = ax^2 + bx + c$, where $x > 0$, $a > 0$, $b > 0$, $c > 0$, then the function reaches

(a) a maximum at $x = \sqrt{\frac{c}{a}}$

(b) a minimum at $x = \sqrt{\frac{c}{a}}$

(c) an inflection point $x = \sqrt{\frac{c}{a}}$

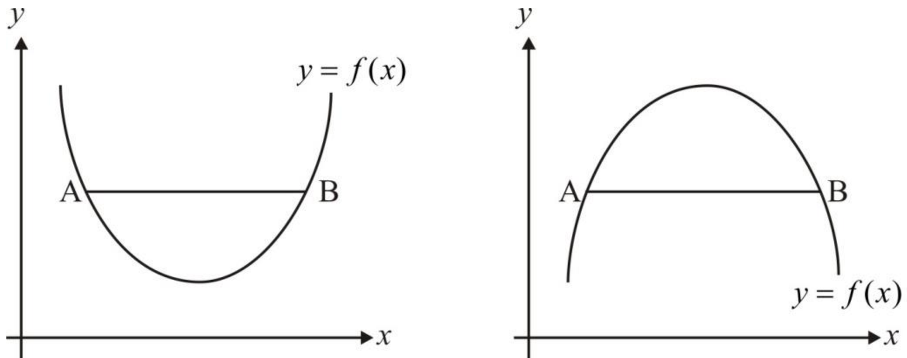
(d) a minimum at $x = \sqrt{\frac{c}{a}}$

13.6 ALTERNATIVE CRITERIA FOR CONCAVITY / CONVEXITY OF A FUNCTION

We will now discuss a derivative-free criterion to identify whether a given function is convex or concave. According to this criterion, a function $f(x)$ is a concave function if the line segment joining any two points on the graph is never above the graph[2]. Similarly, a function

[2] Recall that a line passing through any two points of a given curve is called a secant line. As the distance between the two points is reduced, the secant line tends to be a tangent line.

$f(x)$ is a convex function if the line segment joining any two points on the graph is never below the graph. This is illustrated in Figure 4(a) and 4(b).



Convex Function 4(a) **Concave Functions 4(B)**

Figure

Algebraic Formulation of the Alternative Criteria

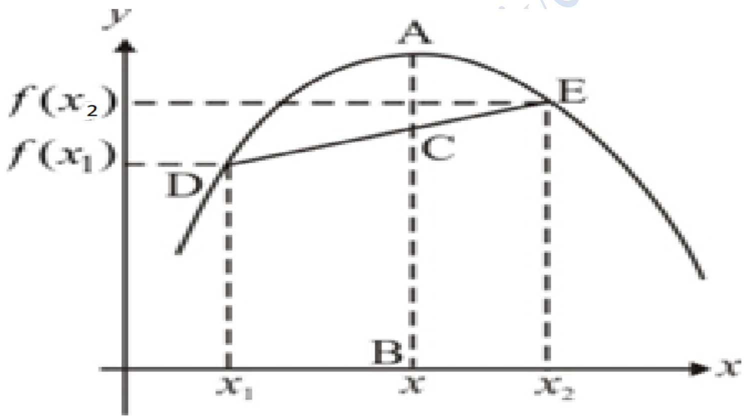


Figure 5

Consider figure 5. Here, any arbitrary point in the interval $[x_1, x_2]$ with $x_1 < x_2$ can be written as:

$$x = (1-\lambda)x_1 + \lambda x_2$$

$$= x_1 + \lambda(x_2 - x_1) \text{ for some } \lambda \in [0, 1] \quad (1)$$

Reason

From (1), $\lambda = \dfrac{x - x_1}{x_2 - x_1}$

Since $x \in [x_1, x_2], \lambda \in [0, 1]$

Now, $(1-\lambda)x_1 + \lambda x_2 = \left(1 - \dfrac{x - x_1}{x_2 - x_1}\right)x_1 + \left(\dfrac{x - x_1}{x_2 - x_1}\right)x_2$

$$= \frac{(x_2 - x_1)x_1 - (x - x_1)(x_1)}{x_2 - x_1} + \frac{(x - x_1)(x_2)}{x_2 - x_1}$$

$$= \frac{(x_2 - x_1)(x_1) + (x - x_1)(x_2 - x_1)}{(x_2 - x_1)}$$

$$= (x_2 - x_1)\frac{(x_1 + x - x_1)}{(x_2 - x_1)} = x$$

Now, we wish to determine the distance CB i.e., the functional value of the line segment at *x*.

Let $CB = y$.

Then, according to the point-point formula of the equation of a line,

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

Putting $x = (1-\lambda)x_1 + \lambda x_2$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}[(1-\lambda)x_1 + \lambda x_2 - x_1]$$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}[x_1 - \lambda x_1 + \lambda x_2 - x_1]$$

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}[\lambda(x_2 - x_1)]$$

$$y - f(x_1) = [\lambda f(x_2) - f(x_1)]$$

$$y = (1-\lambda) f(x_1) + f(x_2)$$ [Line segment CB]

Now, the value of the function at x i.e., segment AB =

$$f(x) = f((1-\lambda)x_1 + \lambda x_2)$$

Hence, it is clear from the figure that the function $f(x)$ is concave in the interval I for all $(x_1, x_2) \in I$ and all $\lambda \in (0, 1)$ if:

$$f((1-\lambda)x_1 + \lambda x_2) \geq (1-\lambda) f(x_1) + \lambda f(x_2)$$

Similarly, a function $f(x)$ is convex in the interval I if

$\forall\ x_1, x_2 \in I$ and all $\lambda \in (0, 1)$

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda) f(x_1) + \lambda(x_2)$$

We also define strict convexity and concavity where the ≤ / ≥ is replaced by < / >.

The function $f(x)$ is strictly concave in the interval I if $\forall (x_1, x_2) \in I$ and $\forall \lambda \in (0,1)$

$$f((1-\lambda)x_1 + \lambda x_2) > (1-\lambda) f(x_1) + \lambda f(x_2)$$

and strictly convex if

$$f((1-\lambda)x_1 + \lambda x_2) < (1-\lambda) f(x_1) + \lambda f(x_2)$$

13.6.1 Jensen's Inequality

According to Jensen's inequality, a function $f(x)$ is concave in the interval I iff the following inequality is satisfied $\forall\ x_1, x_2, ..., x_n \in I$ and $\forall\ \lambda_1, \lambda_2, ..., \lambda_n \geq 0$

with $\lambda_1 + \lambda_2 + ... + \lambda_n = 1$

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + \lambda_n f(x_n)$$

The function is convex iff:

$$f(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + ... + \lambda_n f(x_n)$$

EXAMPLE 4 Prove that the function $y = x^2$ is strictly convex in R using the alternative criteria for convexity.

SOLUTION: $y = x^2$

We know a function $f(x)$ is strictly convex in an interval I if

$$f((1-\lambda)x_1 + \lambda x_2) < (1-\lambda) f(x_1) + \lambda f(x_2)$$

Now, since $f(x) = x^2$

$$f((1-\lambda)x_1 + \lambda x_2) = ((1-\lambda)x_1 + \lambda x_2)^2$$

and

$$f(x_1) = x_1^2$$

$$f(x_2) = x_2^2$$

Hence, from (1)

$$((1-\lambda)x_1 + \lambda x_2)^2 < (1-\lambda)x_1^2 + \lambda x_2^2$$

$$((1-\lambda)x_1 + \lambda x_2)^2 - (1-\lambda)x_1^2 - \lambda x_2^2 < 0 \text{ for convexity}$$

Now,

$$(1-\lambda)^2 x_1^2 + \lambda^2 x_2^2 + 2\lambda(1-\lambda)x_1x_2 - (1-\lambda)x_1^2 - \lambda x_2^2$$

$$= (1-\lambda)x_1^2(1-\lambda-1) + \lambda x_2^2(\lambda-1) + 2\lambda(1-\lambda)x_1x_2$$

$$= (1-\lambda)(-\lambda)x_1^2 + \lambda x_2^2(\lambda-1) + 2\lambda(1-\lambda)x_1x_2$$

$$= \lambda(\lambda-1)x_1^2 + \lambda(\lambda-1)x_2^2 - 2\lambda(\lambda-1)x_1x_2$$

$$= \lambda(\lambda-1)(x_1-x_2)^2 < 0$$

Hence, in this case, $f((1-\lambda)x_1 + \lambda x_2) < (1-\lambda) f(x_1) + \lambda f(x_1)$

Hence, the function is convex.

13.7 SUMMARY

In this lesson, we discussed the concept of concavity/convexity of a function. We discussed how we can determine the curvature of a function by studying the sign of its second derivative. We also learnt about the concept of inflection points and their identification. Finally, the lesson concluded with a discussion on a derivative-free criteria for concavity/convexity of a function.

13.8 GLOSSARY

Concave function: A function $f(x)$ is concave if the line segment joining any two points on the graph is never above the graph.

Convex function: A function $f(x)$ is a convex function if the line segment joining any two points on the graph is never below the graph.

Inflection points: The points at which a function changes its curvature from convex to concave (or vice-versa) are called Inflection Points. Mathematically, if *c* is an interior point in an interval I even which the function is defined and if $f''(c) = 0$ and $f''(x)$ changes sign at *c*, then *c* is an inflection point for *f*.

13.9 ANSWERS TO IN-TEXT QUESTIONS

1. (b)
2. (b)
3. (d)
4. (a)

13.10 TERMINAL QUESTIONS

Q1) Given $f(x) = \frac{2x}{x^2+1}$ show that this curve has 3 points of inflection separated by a maximum and a minimum.

Q2) Determine the concavity / convexity of the following functions:

(i) $f(s) = s^2 e^s$

(ii) $f(x) = 5\sqrt{x} - 10x^{3/2}$

Q3) Using the alternatives criteria for concavity/convexity, prove that the function $f(x) = \alpha x^2 + \beta x + c$, is concave if α ≤ 0 and convex if α ≥0.

Q4) Examine the nature of the cost function $C(q) = a(q-5)^3 + b$ where *q* is the output and *a*, and *b* are positive constants.

13.11 REFERENCES

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